Chapter 2

SCATTERING OF TWO NUCLEONS

This chapter is not meant as a thorough presentation of scattering theory for two nucleons or even more general for two particles since that is already given at many places such as quantum mechanics textbooks and those specializing in scattering processes, for example Ref. [22]. In fact, compact presentations of two nucleon (2N) scattering can be found in Refs. [23, 24, 25]. Hence, the presentation here will be even more compact and this chapter is meant for practical purpose and to give a short summary of necessary formulas. In addition, definitions of some terminologies and quantities used in the next chapters can be found here.

2.1 Kinematics of the Two-Nucleon System in Laboratory and Center of Mass Reference Frames

A proton and a neutron are commonly called nucleon. Though the proton mass $m_p = 938.272$ MeV differs from the neutron mass $m_n = 939.56533$ MeV, this difference is relatively small ($\sim 0.14\%$). Therefore, the 'nucleon mass' m may be given by the average of m_p and m_n.

Let $\mathbf{k}_i$ and $\mathbf{k}'_i$ be the nucleon's momentum in the laboratory reference frame (laboratory frame) in initial and final state, respectively, where $i = 1, 2$ indicates the i^{th} nucleon. The corresponding nonrelativistic energies are denoted by E_i and E'_i, respectively. Assuming nucleon 1 is the projectile and nucleon 2 is the target ($\mathbf{k}_2 = 0$), the momentum situation can be displayed by Fig. 2.1, where θ_{lab} is the scattering angle in the laboratory frame. The figure also shows quantities belonging to the center of mass reference frame (c.m. frame),

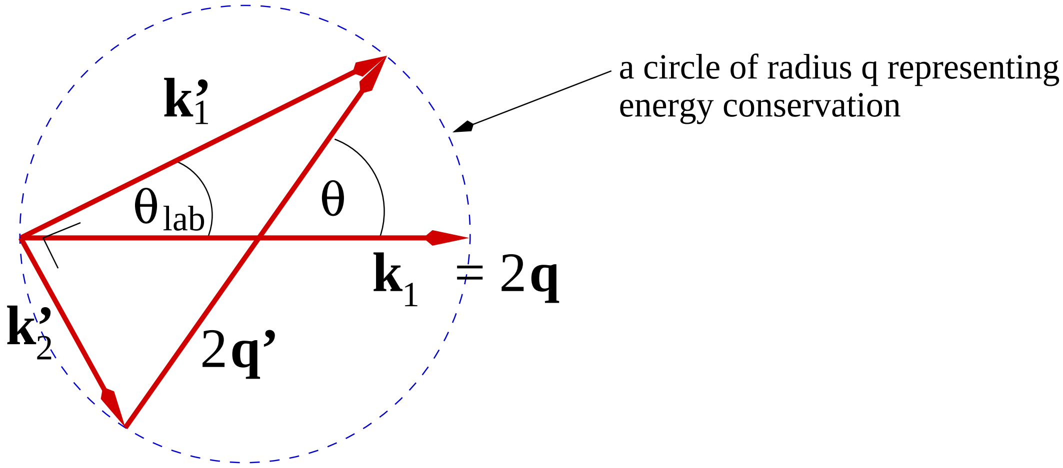

Figure 2.1: The initial and final momenta, both in laboratory and c.m. frames, in a 2N scattering process, where nucleon 1 acts as the projectile and nucleon 2 as the target ($\mathbf{k}_2 = 0$). The circle of radius q represents the energy conservation.

i.e. the scattering angle θ and the relative momentum between the two nucleons in initial and final states, $\mathbf{q} = \frac{1}{2}\mathbf{k}_1$ and $\mathbf{q}' = \frac{1}{2}(\mathbf{k}'_1 - \mathbf{k}'_2)$, respectively. It is clear that $\theta = 2\theta_{lab}$.

The total energy in the laboratory frame (E_{lab}) and that in the c.m. frame (E_{cm}) are

$$E_{lab} = E_1 \quad = \quad E'_1 + E'_2 \tag{2.1}$$

$$E_{lab} = \frac{k_1^2}{2m} \quad = \quad \frac{k_1'^2}{2m} + \frac{k_2'^2}{2m} \tag{2.2}$$

$$E_{cm} = \frac{q^2}{2\mu} \quad = \quad \frac{q^2}{m} = \frac{q'^2}{m}, \tag{2.3}$$

where $\mu = \frac{1}{2}m$ is the reduced mass of the 2N system. E_{cm} together with the energy of motion of the center of mass of the two nucleons sum up to E_{lab} and consequently we can get the relation between E_{lab} and E_{cm}

$$\begin{aligned} E_{lab} &= \frac{(\mathbf{k}'_1 + \mathbf{k}'_2)^2}{4m} + E_{cm} \\ &= \frac{k_1^2}{4m} + E_{cm} \\ &= \frac{1}{2}E_{lab} + E_{cm} \\ &= 2E_{cm}, \end{aligned} \tag{2.4}$$

which can also be directly seen from the fact that $\mathbf{k}_1 = 2\mathbf{q}$. Note that this relation between E_{lab} and E_{cm} is correct if one of the two nucleons is initially (or finally) at rest.

2.2 Scattering Matrix and Lippmann-Schwinger Equation

The essential information of a nucleon-nucleon (NN) scattering process is contained in the scattering matrix. There are T-matrix, S-matrix, M-matrix and these matrices are related to each other as

$$\begin{aligned} S &= 1 - 2\pi i\delta(E' - E)T && (2.5) \\ M &= -\mu(2\pi)^2 T. && (2.6) \end{aligned}$$

The delta function in the expression for the S-matrix indicates that the S-matrix is an on-the-energy-shell (on-shell) quantity whereas the other two scattering matrices are not affected by this restriction and therefore have off-shell as well as on-shell properties. We solve for the T-matrix in our NN scattering calculations and later use it as input for our 3N calculations, where the T-matrix appears as an off-shell quantity.

The T-matrix obeys the equation

$$T = V + VG_0T, \tag{2.7}$$

which is the Lippmann-Schwinger Equation (LSE) for the T-matrix. V is the matrix operator of the NN potential, $G_0(z) = (z - H_0)^{-1}$ is the free propagator with H_0 being the free Hamiltonian and z a complex number. The scattering wave is spreading out from the scattering center, and for an outgoing wave the corresponding free propagator is $G_0^+(E) \equiv \lim_{\epsilon\to 0} G_0(E + i\epsilon)$, where E is the energy at which the scattering occurs and the limit can be understood as to bring z close to the physical spectrum of H_0.

The T-matrix element is defined as

$$T(\mathbf{q}', \alpha'; \mathbf{q}, \alpha) \equiv \langle \mathbf{q}', \alpha' | T | \mathbf{q}, \alpha \rangle, \tag{2.8}$$

with α, α' being the discrete quantum numbers considered, like spin and isospin, and $|\mathbf{q}, \alpha\rangle$, $|\mathbf{q}', \alpha'\rangle$ representing the initial, final state of the 2N system, respectively. A similar definition applies also to the NN potential matrix element

$$V(\mathbf{q}', \alpha'; \mathbf{q}, \alpha) \equiv \langle \mathbf{q}', \alpha' | V | \mathbf{q}, \alpha \rangle. \tag{2.9}$$

With the 2N states $|\mathbf{q}, \alpha\rangle$ being complete

$$\sum_\alpha \int d\mathbf{q}\, |\mathbf{q}, \alpha\rangle \langle \mathbf{q}, \alpha| = 1, \tag{2.10}$$

it is straightforward that the LSE for the T-matrix element, which is the main equation in the calculations, is given by

$$T(\mathbf{q}',\alpha';\mathbf{q},\alpha) = V(\mathbf{q}',\alpha';\mathbf{q},\alpha) + \sum_{\alpha''}\int d\mathbf{q}'' V(\mathbf{q}',\alpha';\mathbf{q}'',\alpha'')G_0^+(E_q)T(\mathbf{q}'',\alpha'';\mathbf{q},\alpha), \quad (2.11)$$

with

$$G_0^+(E_q) = \lim_{\epsilon\to 0}\frac{1}{E_q + i\epsilon - E_{q''}} \qquad E_q \equiv \frac{q^2}{m} \qquad E_{q''} \equiv \frac{q''^2}{m}. \quad (2.12)$$

2.3 Cross Section and Spin Observables

Here we specify the quantum number α in the 2N state $|\mathbf{q},\alpha\rangle$ as the magnetic spin quantum numbers of both nucleons

$$|\mathbf{q},\alpha\rangle = |\mathbf{q}, m_{s1}m_{s2}\rangle\,, \quad (2.13)$$

with $m_{si} = \pm\frac{1}{2}$ (i = 1, 2). Thus, there are four spin states which constitute a complete basis, in which any spin state of the two nucleons can be given. A general pure state $|\mathbf{q},n\rangle$ can be written as

$$|\mathbf{q},n\rangle = \sum_{m_{s1},m_{s2}=-\frac{1}{2}}^{\frac{1}{2}} a^{(n)}(m_{s1},m_{s2})\,|\mathbf{q},m_{s1}m_{s2}\rangle\,. \quad (2.14)$$

With regard to spin the state $|\mathbf{q},n\rangle$ is a vector of four components and the T-matrix element given in Eq. (2.8) is a 4 x 4 matrix. The general spin operator for such a state is also a 4 x 4 matrix and may be chosen as a product of two 2 x 2 matrices

$$\sigma_\mu^{(1)}\sigma_\nu^{(2)} \equiv \sigma_\mu^{(1)}\otimes\sigma_\nu^{(2)}, \qquad (\mu,\nu = 0,1,2,3), \quad (2.15)$$

with σ_0 and σ_i (i = 1,2,3) being a matrix of one and the Pauli matrices, respectively:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.16)$$

and the upper indices 1, 2 denoting the nucleon on the state of which the σ_μ operator works.

In experiments we deal not only with two nucleons but many more in the beam and the target. Therefore, the state is a mixed state of pure 2N states as given in of Eq. (2.14), and the expectation value of an observable $\langle O\rangle$ is calculated by mean of a density matrix ρ

$$\rho \equiv \sum_n |n\rangle\, p_n\, \langle n|\,, \quad (2.17)$$

where p_n is the normalized probability of the n^{th} pure spin state according to Eq. (2.14)

$$|n\rangle \equiv \sum_{m_{s1},m_{s2}=-\frac{1}{2}}^{\frac{1}{2}} a^{(n)}(m_{s1},m_{s2})\,|m_{s1}m_{s2}\rangle\,. \tag{2.18}$$

For instance, in the final state:

$$\langle O\rangle = \frac{Tr\,\{\rho_f O\}}{Tr\,\{\rho_f\}}, \tag{2.19}$$

with

$$\begin{aligned}\rho_f &= \text{final density matrix}\\ &= M\rho_i M^\dagger \\ \rho_i &= \text{initial density matrix.}\end{aligned} \tag{2.20}$$

Using Eq. (2.19) one derives the expression for the expectation value of a general spin observable $\left\langle\sigma_\mu^{(1)}\sigma_\nu^{(2)}\right\rangle_f$ in the final state in relation to the values $\left\langle\sigma_\alpha^{(1)}\sigma_\beta^{(2)}\right\rangle_i$ in the initial state

$$I\left\langle\sigma_\mu^{(1)}\sigma_\nu^{(2)}\right\rangle_f = \frac{1}{4}\sum_{\alpha,\beta}\left\langle\sigma_\alpha^{(1)}\sigma_\beta^{(2)}\right\rangle_i Tr\left\{M\sigma_\alpha^{(1)}\sigma_\beta^{(2)}M^\dagger\sigma_\mu^{(1)}\sigma_\nu^{(2)}\right\}, \tag{2.21}$$

where I is the differential cross section summed over all possible final spin states

$$I = \sum_j \frac{d\sigma_j}{d\Omega} = \frac{Tr\,\{\rho_f\}}{Tr\,\{\rho_i\}} = \frac{1}{4}\sum_{\alpha,\beta}\left\langle\sigma_\alpha^{(1)}\sigma_\beta^{(2)}\right\rangle_i Tr\left\{M\sigma_\alpha^{(1)}\sigma_\beta^{(2)}M^\dagger\right\} \tag{2.22}$$

(in the last equality Eq. (2.21) is applied again).

The simplest case is if the beam and target are unpolarized and no spin measurements in the final state are made. In this case one measures the spin averaged differential cross section

$$I_0 = \frac{1}{4}Tr\left\{MM^\dagger\right\}. \tag{2.23}$$

The spin projections on a certain axis must be specified and therefore unit vectors are needed. Since there are two reference frames - laboratory and c.m. frames - two sets of unit vectors are defined, one set for each frame. But as can be checked in Ref. [25] for the 2N system the two sets are the same:

$$\text{unit vectors for the initial state} \;:\; \begin{cases}\text{c.m. frame} \;:\; \hat{\mathbf{q}},\, \hat{\mathbf{N}},\, \hat{\mathbf{N}}\times\hat{\mathbf{q}}\\ \text{laboratory frame} \;:\; \hat{\mathbf{l}},\, \hat{\mathbf{n}},\, \hat{\mathbf{s}}\end{cases} \tag{2.24}$$

$$\text{unit vectors for the final state} \;:\; \begin{cases}\text{c.m. frame} \;:\; \hat{\mathbf{P}},\, \hat{\mathbf{N}},\, \hat{\mathbf{K}}\\ \text{laboratory frame} \;:\; \hat{\mathbf{l}}',\, \hat{\mathbf{n}}',\, \hat{\mathbf{s}}'\end{cases} \tag{2.25}$$

with

$$\begin{aligned}
\hat{\mathbf{n}} &= \hat{\mathbf{n}}' \equiv \frac{\hat{\mathbf{k}}_1 \times \hat{\mathbf{k}}_1'}{|\hat{\mathbf{k}}_1 \times \hat{\mathbf{k}}_1'|} = \hat{\mathbf{N}} \equiv \frac{\mathbf{q} \times \mathbf{q}'}{|\mathbf{q} \times \mathbf{q}'|} && (2.26)\\
\hat{\mathbf{l}} &\equiv \hat{\mathbf{k}}_1 = \hat{\mathbf{q}} && (2.27)\\
\hat{\mathbf{s}} &\equiv \hat{\mathbf{n}} \times \hat{\mathbf{l}} = \hat{\mathbf{N}} \times \hat{\mathbf{q}} && (2.28)\\
\hat{\mathbf{l}}' &\equiv \hat{\mathbf{k}}_1' = \hat{\mathbf{P}} \equiv \frac{\mathbf{q} + \mathbf{q}'}{|\mathbf{q} + \mathbf{q}'|} && (2.29)\\
\hat{\mathbf{s}}' &\equiv \hat{\mathbf{n}}' \times \hat{\mathbf{l}}' = \hat{\mathbf{K}} \equiv \frac{\mathbf{q}' - \mathbf{q}}{|\mathbf{q}' - \mathbf{q}|}. && (2.30)
\end{aligned}$$

In connection with a Cartesian coordinate system the beam's momentum $\mathbf{k}_1$ is set typically to point along the positive z-axis and the scattered nucleon's momentum $\mathbf{k}_1'$ is in the xz-plane. Thus, the scattering takes places in the xz-plane and the unit vectors are

$$\begin{aligned}
&\hat{\mathbf{l}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \hat{\mathbf{s}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{\mathbf{n}} = \hat{\mathbf{n}}' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\
&\hat{\mathbf{l}}' = \begin{pmatrix} \sin\theta_{lab} \\ 0 \\ \cos\theta_{lab} \end{pmatrix}, \qquad \hat{\mathbf{s}}' = \begin{pmatrix} \cos\theta_{lab} \\ 0 \\ -\sin\theta_{lab} \end{pmatrix}.
\end{aligned} \tag{2.31}$$

According to Eq. (2.21) there seems to be 16 x 16 = 256 possible initial-to-final spin transitions in a NN scattering process. Rotational, parity, time-reversal and isospin invariances (the last one together with parity invariance lead to spin invariance), however, forbid many transitions and moreover cause some permitted transitions to be related to each other. Under these invariances the scattering matrix M can be expressed in terms of a few parameters called Wolfenstein parameters [26, 23] (a, c, m, g, h), which depend on the magnitudes q' of final and q of initial relative momenta as well as the angle between the two momenta $\mathbf{q}'$ and $\mathbf{q}$

$$\begin{aligned}
M &= a + c(\boldsymbol{\sigma}^{(1)} + \boldsymbol{\sigma}^{(2)}) \cdot \hat{\mathbf{N}} + m(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{N}})(\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{N}}) \\
&\quad + (g+h)(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{P}})(\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{P}}) + (g-h)(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{K}})(\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{K}})
\end{aligned} \tag{2.32}$$

$$\begin{aligned}
a &= \frac{1}{4} T_r\{M\} && (2.33)\\
c &= \frac{1}{8} Tr\{M\sigma_y^{(1)} + M\sigma_y^{(2)}\} && (2.34)\\
m &= \frac{1}{4} Tr\{M\sigma_y^{(1)}\sigma_y^{(2)}\} && (2.35)
\end{aligned}$$

$$g = \frac{1}{8}Tr\{M\sigma_x^{(1)}\sigma_x^{(2)} + M\sigma_z^{(1)}\sigma_z^{(2)}\} \tag{2.36}$$

$$h = \frac{1}{8}Tr\{[-M\sigma_x^{(1)}\sigma_x^{(2)} + M\sigma_z^{(1)}\sigma_z^{(2)}]\cos\theta + [M\sigma_x^{(1)}\sigma_z^{(2)} + M\sigma_z^{(1)}\sigma_x^{(2)}]\sin\theta\} \tag{2.37}$$

Note that these expressions for the Wolfenstein parameters are for the chosen xz-scattering frame, see Eq. (2.31). The NN scattering observables can be calculated using M directly or the Wolfenstein parameters.

Finally, we close this chapter by showing briefly seven typical types of experiments and the corresponding spin observables. Comprehensive descriptions of these experiments can be found in Ref. [23]. The experiments are denoted by the reactions as

$$1.\ \ N2(N1,N1)N2 \quad 2.\ \ N2(N1,\vec{N}1)N2 \quad 3.\ \ N2(\vec{N}1,N1)N2 \quad 4.\ \ N2(\vec{N}1,\vec{N}1)N2$$
$$5.\ \ N2(\vec{N}1,N1)\vec{N}2 \quad 6.\ \ N2(N1,\vec{N}1)\vec{N}2 \quad 7.\ \ \vec{N}2(\vec{N}1,N1)N2,$$

where $N1$ and $N2$ stand for nucleon 1 (the projectile) and nucleon 2 (the target), respectively, the little arrows over $N1$ or $N2$ mean that the corresponding nucleon is polarized or that the polarization of that nucleon is measured. Let us take for example the fifth experiment: $N2(\vec{N}1,N1)\vec{N}2$. This reaction means that a polarized projectile ($\vec{N}1$) is directed to an unpolarized target ($N2$) and finally the polarization of the recoil nucleon ($\vec{N}2$) is measured. The polarization of the scattered nucleon ($N1$) is not measured. Note that processes 4 and 5 are only distinguishable for a np system.

In the first experiment the beam and target are unpolarized and no spin measurement on the outgoing nucleons are made. One measures only the spin averaged cross section

$$\begin{aligned} I_0 &= \frac{1}{4}Tr\left\{MM^\dagger\right\} \\ &= |a|^2 + |m|^2 + 2|c|^2 + 2|g|^2 + 2|h|^2. \end{aligned} \tag{2.38}$$

In the second experiment the beam and target are unpolarized. The polarization of the scattered nucleon is of interest and therefore after the process one measures the spin direction of this nucleon. According to the general formula for spin observables (Eq. (2.21)) the polarization $\mathbf{P}_0 = \left\langle \boldsymbol{\sigma}^{(1)} \right\rangle = \left\langle \boldsymbol{\sigma}^{(1)}\sigma_0^{(2)} \right\rangle$ of the scattered nucleon is

$$\begin{aligned} \mathbf{P}_0 &= \frac{1}{4I_0}Tr\left\{MM^\dagger\boldsymbol{\sigma}^{(1)}\right\} \\ &= \hat{\mathbf{n}}\frac{1}{4I_0}Tr\left\{MM^\dagger\sigma_n^{(1)}\right\} \\ &= \hat{\mathbf{n}}\frac{2Re\{(a+m)c^*\}}{I_0}, \end{aligned} \tag{2.39}$$

where I_0 is the spin averaged cross section given in Eq. (2.38). Parity invariance affects the process such that the polarization must be normal to the scattering plane.

The third experiment is to measure the asymmetry A_{LR} defined as

$$A_{LR} \equiv \frac{I_L - I_R}{I_L + I_R}, \tag{2.40}$$

where $I_L = I(\theta, \phi)$ and $I_R = I(\theta, \phi + \pi)$ are the left-scattering and right-scattering cross sections, respectively. A polarized beam is directed to an unpolarized target. Due to parity invariance a contribution to the cross section arises only if the polarization is normal to the scattering plane. The cross section is

$$\begin{aligned} I &= \frac{1}{4}\sum_{\alpha=0}^{3} \left\langle \sigma_\alpha^{(1)} \right\rangle_i Tr\left\{ M\sigma_\alpha^{(1)} M^\dagger \right\} \\ &= I_0 + \frac{1}{4}\mathbf{P}_i \cdot \hat{\mathbf{n}} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{n}}) M^\dagger \right\} \end{aligned} \tag{2.41}$$

and the left- and right-scattering cross sections are

$$I_L = I_0 + \frac{1}{4}\mathbf{P}_i \cdot \hat{\mathbf{n}} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{n}}) M^\dagger \right\} \tag{2.42}$$

$$I_R = I_0 - \frac{1}{4}\mathbf{P}_i \cdot \hat{\mathbf{n}} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{n}}) M^\dagger \right\}. \tag{2.43}$$

Therefore,

$$\begin{aligned} A_{LR} &= \frac{\mathbf{P}_i \cdot \hat{\mathbf{n}} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{n}}) M^\dagger \right\}}{4I_0} \\ &= \mathbf{P}_i \cdot \hat{\mathbf{n}} A_n, \end{aligned} \tag{2.44}$$

with

$$\begin{aligned} A_n &= \frac{1}{4I_0} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{n}}) M^\dagger \right\} \\ &= \frac{2Re\{(a+m)c^*\}}{I_0} \\ &= P_0. \end{aligned} \tag{2.45}$$

This quantity A_n called analyzing power is often denoted by A_y, since $\hat{\mathbf{n}} = \hat{y}$ for the typical scattering frame given in Eq. (2.31).

In experiment 4 one starts with a polarized beam and an unpolarized target and finally measures the polarization of the scattered nucleon, $\mathbf{P}_f = \left\langle \boldsymbol{\sigma}^{(1)} \right\rangle$

$$\begin{aligned} I\mathbf{P}_f &= \frac{1}{4}\sum_{\alpha=0}^{3} \left\langle \sigma_\alpha^{(1)} \right\rangle_i Tr\left\{ M\sigma_\alpha^{(1)} M^\dagger \boldsymbol{\sigma}^{(1)} \right\} \\ &= I_0\mathbf{P}_0 + \frac{1}{4}\mathbf{P}_i \cdot Tr\left\{ M\boldsymbol{\sigma}^{(1)} M^\dagger \boldsymbol{\sigma}^{(1)} \right\} \\ &= I_0 \Big\{ \hat{\mathbf{n}} \left[P_0 + D(\mathbf{P}_i \cdot \hat{\mathbf{n}}) \right] + \hat{\mathbf{l}}' \left[A'(\mathbf{P}_i \cdot \hat{\mathbf{l}}) + R'(\mathbf{P}_i \cdot \hat{\mathbf{s}}) \right] \\ &\qquad + \hat{\mathbf{s}}' \left[A(\mathbf{P}_i \cdot \hat{\mathbf{l}}) + R(\mathbf{P}_i \cdot \hat{\mathbf{s}}) \right] \Big\}. \end{aligned} \tag{2.46}$$

Here we meet other spin observables, summarized in the depolarization tensor D_{ij}, which is defined as

$$I_0 D_{ij} \equiv \frac{1}{4} Tr \left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{j}}) M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{i}}) \right\}, \tag{2.47}$$

and the observables D, R, R', A, A' appearing in the polarization $\vec{P}_f$ are

$$\begin{aligned}
I_0 D \equiv I_0 D_{nn} &= \frac{1}{4} Tr \left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{n}}) M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{n}}) \right\} \\
&= |a|^2 + |m|^2 + 2|c|^2 - 2|g|^2 - 2|h|^2 && (2.48) \\
I_0 R \equiv I_0 D_{s's} &= \frac{1}{4} Tr \left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{s}}) M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{s}}') \right\} \\
&= (|a|^2 - |m|^2 - 4Re\{gh^*\}) \cos\frac{\theta}{2} - 2Im\{(a-m)^* c\} \sin\frac{\theta}{2} && (2.49) \\
I_0 R' \equiv I_0 D_{l's} &= \frac{1}{4} Tr \left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{s}}) M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{l}}') \right\} \\
&= (|a|^2 - |m|^2 + 4Re\{gh^*\}) \sin\frac{\theta}{2} + 2Im\{(a-m)^* c\} \cos\frac{\theta}{2} && (2.50) \\
I_0 A \equiv I_0 D_{s'l} &= \frac{1}{4} Tr \left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{l}}) M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{s}}') \right\} \\
&= -(|a|^2 - |m|^2 - 4Re\{gh^*\}) \sin\frac{\theta}{2} - 2Im\{(a-m)^* c\} \cos\frac{\theta}{2} && (2.51) \\
I_0 A' \equiv I_0 D_{l'l} &= \frac{1}{4} Tr \left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{l}}) M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{l}}') \right\} \\
&= (|a|^2 - |m|^2 + 4Re\{gh^*\}) \cos\frac{\theta}{2} - 2Im\{(a-m)^* c\} \sin\frac{\theta}{2}. && (2.52)
\end{aligned}$$

Experiment 5 is similar to experiment 4 and can be distinguished only in a np system. One starts with a polarized beam and an unpolarized target but finally one measures the polarization of the recoil nucleon $\mathbf{P}_f = \left\langle \boldsymbol{\sigma}^{(2)} \right\rangle$

$$\begin{aligned}
I\mathbf{P}_f &= \frac{1}{4} \sum_{\alpha=0}^{3} \left\langle \sigma_\alpha^{(1)} \right\rangle_i Tr \left\{ M \sigma_\alpha^{(1)} M^\dagger \boldsymbol{\sigma}^{(2)} \right\} \\
&= I_0 \mathbf{P}_0 + \frac{1}{4} \mathbf{P}_i \cdot Tr \left\{ M \boldsymbol{\sigma}^{(1)} M^\dagger \boldsymbol{\sigma}^{(2)} \right\} \\
&= I_0 \left\{ \hat{\mathbf{n}} \left[P_0 + D_t (\mathbf{P}_i \cdot \hat{\mathbf{n}}) \right] + \hat{\mathbf{l}}' \left[A_t (\mathbf{P}_i \cdot \hat{\mathbf{l}}) + R_t (\mathbf{P}_i \cdot \hat{\mathbf{s}}) \right] \right. \\
&\qquad \left. + \, \hat{\mathbf{s}}' \left[A_t' (\mathbf{P}_i \cdot \hat{\mathbf{l}}) + R_t' (\mathbf{P}_i \cdot \hat{\mathbf{s}}) \right] \right\} && (2.53)
\end{aligned}$$

$$P_0 = \frac{1}{4I_0} Tr \left\{ M M^\dagger (\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{n}}) \right\} = \frac{2Re\{(a+m)c^*\}}{I_0}. \tag{2.54}$$

Here we have again new spin observables, summarized in the polarization-transfer tensor K_{ij}, which is defined as

$$I_0 K_{ij} \equiv \frac{1}{4} Tr \left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{j}}) M^\dagger (\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{i}}) \right\}, \tag{2.55}$$

and the observables $D_t, R_t, R'_t, A_t, A'_t$ appearing in the polarization $\mathbf{P}_f$ are

$$\begin{aligned} I_0 D_t \equiv I_0 K_{nn} &= \frac{1}{4} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{n}}) M^\dagger (\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{n}}) \right\} \\ &= 2(Re\{am^*\} + |c|^2 + |g|^2 - |h|^2) \qquad (2.56) \\ I_0 R_t \equiv I_0 K_{l's} &= \frac{1}{4} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{s}}) M^\dagger (\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{l}}') \right\} \\ &= 2Re\{(a+m)g^* + (a-m)h^*\} \sin\frac{\theta}{2} + 4Im\{cg^*\}\cos\frac{\theta}{2} \qquad (2.57) \\ I_0 R'_t \equiv I_0 K_{s's} &= \frac{1}{4} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{s}}) M^\dagger (\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{s}}') \right\} \\ &= 2Re\{(a+m)g^* - (a-m)h^*\} \cos\frac{\theta}{2} - 4Im\{cg^*\}\sin\frac{\theta}{2} \qquad (2.58) \\ I_0 A_t \equiv -I_0 K_{l'l} &= -\frac{1}{4} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{l}}) M^\dagger (\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{l}}') \right\} \\ &= -2Re\{(a+m)g^* + (a-m)h^*\} \cos\frac{\theta}{2} + 4Im\{cg^*\}\sin\frac{\theta}{2} \qquad (2.59) \\ I_0 A'_t \equiv -I_0 K_{s'l} &= -\frac{1}{4} Tr\left\{ M(\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{l}}) M^\dagger (\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{s}}') \right\} \\ &= 2Re\{(a+m)g^* - (a-m)h^*\} \sin\frac{\theta}{2} + 4Im\{cg^*\}\cos\frac{\theta}{2}. \qquad (2.60) \end{aligned}$$

Note the minus sign in the definitions for A_t and A'_t. These are the definitions given in Center for Nuclear Studies Data Analysis Center (CNS DAC, http://gwdac.phys.gwu.edu/). We take these definitions since later we compare with experimental data from this site. In Ref. [23] the definitions for A_t and A'_t have the opposite sign. In case of identical particles these expressions are the same as the ones given in Eqs. (2.48)-(2.52) if one replaces θ by $\pi - \theta$ (see for instance [23]).

In experiment 6 the beam and target are unpolarized. In the final state the spins of the two outgoing nucleons are simultaneously measured

$$\begin{aligned} I\left\langle \boldsymbol{\sigma}^{(1)}\boldsymbol{\sigma}^{(2)} \right\rangle_f &= \frac{1}{4} Tr\left\{ M M^\dagger \boldsymbol{\sigma}^{(1)} \boldsymbol{\sigma}^{(2)} \right\} \\ &= I_0 \left(C_{NN}\hat{\mathbf{N}}\hat{\mathbf{N}} + C_{PP}\hat{\mathbf{P}}\hat{\mathbf{P}} + C_{KK}\hat{\mathbf{K}}\hat{\mathbf{K}} + C_{KP}(\hat{\mathbf{P}}\hat{\mathbf{K}} + \hat{\mathbf{K}}\hat{\mathbf{P}}) \right). \qquad (2.61) \end{aligned}$$

C_{ij} is called the spin correlation parameter and is defined as

$$I_0 C_{ij} \equiv \frac{1}{4} Tr\left\{ M M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{i}})(\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{j}}) \right\}. \qquad (2.62)$$

Accordingly, $C_{NN}, C_{PP}, C_{KK}, C_{KP}$ are

$$\begin{aligned} I_0 C_{NN} &= \frac{1}{4} Tr\left\{ M M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{N}})(\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{N}}) \right\} = 2(Re\{am^*\} + |c|^2 - |g|^2 + |h|^2) \qquad (2.63) \\ I_0 C_{PP} &= \frac{1}{4} Tr\left\{ M M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{P}})(\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{P}}) \right\} = 2Re\{(a-m)g^* + (a+m)h^*\} \qquad (2.64) \\ I_0 C_{KK} &= \frac{1}{4} Tr\left\{ M M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{K}})(\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{K}}) \right\} = 2Re\{(a-m)g^* - (a+m)h^*\} \qquad (2.65) \\ I_0 C_{KP} &= \frac{1}{4} Tr\left\{ M M^\dagger (\boldsymbol{\sigma}^{(1)} \cdot \hat{\mathbf{K}})(\boldsymbol{\sigma}^{(2)} \cdot \hat{\mathbf{P}}) \right\} = -4Im\{ch^*\}. \qquad (2.66) \end{aligned}$$

It can be shown that $C_{PK} = C_{KP}$.

In the last experiment both the beam and target are polarized and no spin measurements are made in the final state. One measures the cross section

$$\begin{aligned} I &= \frac{1}{4}\sum_{\alpha,\beta}\left\langle \sigma_\alpha^{(1)}\sigma_\beta^{(2)}\right\rangle_i Tr\left\{M\sigma_\alpha^{(1)}\sigma_\beta^{(2)}M^\dagger\right\} \\ &= I_0\left(1 + 2P_{iy}A_y + P_{ixx}A_{xx} + P_{iyy}A_{yy} + P_{izz}A_{zz} - 2P_{ixz}A_{zx}\right). \end{aligned} \tag{2.67}$$

The indices are for the scattering frame given in Eq. (2.31). $P_{iy} = \left\langle \sigma_y^{(1)}\right\rangle_i = \left\langle \sigma_y^{(2)}\right\rangle_i$ and $P_{ikl} = \left\langle \sigma_k^{(1)}\sigma_l^{(2)}\right\rangle_i$ are the polarization and tensor polarization in initial state, respectively. A_y is the already shown analyzing power. The other observables are the spin correlation parameters A_{ij}'s, which are also called tensor analyzing powers defined as

$$A_{ij} \equiv \frac{1}{4I_0}Tr\left\{M(\boldsymbol{\sigma}^{(1)}\cdot\hat{\mathbf{i}})(\boldsymbol{\sigma}^{(2)}\cdot\hat{\mathbf{j}})M^\dagger\right\}. \tag{2.68}$$

Accordingly, $A_{xx}, A_{yy}, A_{zz}, A_{zx}$ are

$$\begin{aligned} I_0A_{xx} \equiv I_0A_{ss} &= \frac{1}{4}Tr\left\{M(\boldsymbol{\sigma}^{(1)}\cdot\hat{\mathbf{s}})(\boldsymbol{\sigma}^{(2)}\cdot\hat{\mathbf{s}})M^\dagger\right\} \\ &= 2Re\{(a-m)g^* - (a+m)h^*\cos\theta\} + 4Im\{ch^*\}\sin\theta \qquad (2.69) \\ I_0A_{yy} \equiv I_0A_{nn} &= \frac{1}{4}Tr\left\{M(\boldsymbol{\sigma}^{(1)}\cdot\hat{\mathbf{n}})(\boldsymbol{\sigma}^{(2)}\cdot\hat{\mathbf{n}})M^\dagger\right\} \\ &= 2(Re\{am^*\} + |c|^2 - |g|^2 + |h|^2) = I_0C_{NN} \qquad (2.70) \\ I_0A_{zz} \equiv I_0A_{ll} &= \frac{1}{4}Tr\left\{M(\boldsymbol{\sigma}^{(1)}\cdot\hat{\mathbf{l}})(\boldsymbol{\sigma}^{(2)}\cdot\hat{\mathbf{l}})M^\dagger\right\} \\ &= 2Re\{(a-m)g^* + (a+m)h^*\cos\theta\} - 4Im\{ch^*\}\sin\theta \qquad (2.71) \\ I_0A_{zx} \equiv -I_0A_{ls} &= -\frac{1}{4}Tr\left\{M(\boldsymbol{\sigma}^{(1)}\cdot\hat{\mathbf{l}})(\boldsymbol{\sigma}^{(2)}\cdot\hat{\mathbf{s}})M^\dagger\right\} \\ &= -2Re\{(a+m)h^*\}\sin\theta - 4Im\{ch^*\}\cos\theta \qquad (2.72) \end{aligned}$$

It can be shown that $A_{zx} = A_{xz}$. Again, note the minus sign in the definition for A_{zx}, which is taken from CNS DAC. In Ref. [23] the definition for A_{zx} has the opposite sign.