

Relativistic Scattering

1 Kinematics

- relasi relativistik momentum \mathbf{k} dan energi E :

$$E^2 = m^2 + \mathbf{k}^2, \quad (1)$$

dengan m massa diam.

- relasi relativistik energi kinetik E_k dan energi E :

$$E = m + E_k \quad (2)$$

- kecepatan \mathbf{v} :

$$\mathbf{v} = \frac{\mathbf{k}}{E} \quad (3)$$

- momentum-4:

$$\begin{aligned} k^\mu &= (k_0, \mathbf{k}) = (E, \mathbf{k}) \\ k_\mu &= (k_0, -\mathbf{k}) = (E, -\mathbf{k}) \\ k^2 &= k^\mu k_\mu = E^2 - \mathbf{k}^2 = m^2 \end{aligned} \quad (4)$$

- relasi momentum dan energi antar dua kerangka acuan, k^μ di kerangka lama dan p^μ di kerangka baru:

$$\begin{aligned} p^\mu &= L(\mathbf{v})k^\mu \\ p_0 &= \gamma(k_0 - \mathbf{k} \cdot \mathbf{v}) \\ \mathbf{p} &= \mathbf{k} + (\gamma - 1)(\mathbf{k} \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} - \gamma k_0 \mathbf{v}, \end{aligned} \quad (5)$$

dengan $L(\mathbf{v})$ transformasi Lorentz, \mathbf{v} kecepatan kerangka acuan baru relatif terhadap kerangka acuan lama, dan $\gamma = (1 - v^2)^{-\frac{1}{2}}$.

- dalam kerangka acuan laboratorium (lab.); partikel 1 proyektil (mis. kaon) dan partikel 2 target diam (mis. nukleon):

$$\begin{aligned} k_1^\mu &= (E_{1,lab}, \mathbf{k}_1) & k_1'^\mu &= (E'_{1,lab}, \mathbf{k}'_1) \\ k_2^\mu &= (E_{2,lab}, \mathbf{k}_2) = (m_2, 0) & k_2'^\mu &= (E'_{2,lab}, \mathbf{k}'_2) \end{aligned} \quad (6)$$

$$E_{lab} = E_{1,lab} + E_{2,lab} = E_{1,lab} + m_2 = E_{1k,lab} + m_1 + m_2 = E'_{1,lab} + E'_{2,lab} = E'_{lab} \quad (7)$$

$$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_1 = \mathbf{k}'_1 + \mathbf{k}'_2 = \mathbf{K}'$$

$$k^\mu = k_1^\mu + k_2^\mu = (E_{lab}, \mathbf{K}) = (E_{1,lab} + m_2, \mathbf{k}_1) = (E_{1k,lab} + m_1 + m_2, \mathbf{k}_1)$$

$$= k'_1^\mu + k'_2^\mu = (E_{lab}, \mathbf{K}) = (E'_{1,lab} + E'_{2,lab}, \mathbf{k}'_1 + \mathbf{k}'_2)$$

$$k^\mu k_\mu = M_0^2 \quad (M_0 = \text{massa invarian sistem}) \quad (8)$$

$$= E_{lab}^2 - \mathbf{K}^2 = (E_{1,lab} + m_2)^2 - \mathbf{k}_1^2 = E_{1,lab}^2 + m_2^2 + 2m_2 E_{1,lab} - \mathbf{k}_1^2$$

$$= m_1^2 + m_2^2 + 2m_2 E_{1,lab}$$

- dalam kerangka acuan pusat massa (p.m.):

$$\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p} \quad \mathbf{p}'_1 = -\mathbf{p}'_2 = \mathbf{p}' = p\hat{\mathbf{p}}' \quad (9)$$

$$E'_1 = E_1 = \sqrt{m_1^2 + \mathbf{p}^2} \quad E'_2 = E_2 = \sqrt{m_2^2 + \mathbf{p}^2}$$

$$p_1^\mu = (E_1, \mathbf{p}_1) = (E_1, \mathbf{p}) \quad p'_1^\mu = (E_1, \mathbf{p}'_1) = (E_1, \mathbf{p}') \quad (10)$$

$$p_2^\mu = (E_2, \mathbf{p}_2) = (E_2, -\mathbf{p}) \quad p'_2^\mu = (E_2, \mathbf{p}'_2) = (E_2, -\mathbf{p}')$$

$$E = E_1 + E_2 = E'_1 + E'_2 = E' = M_0 \quad (\text{lihat bukti di bawah}) \quad (11)$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}'_1 + \mathbf{p}'_2 = \mathbf{P}' = 0$$

$$p^\mu = p_1^\mu + p_2^\mu = (E_1 + E_2, 0) = (E, 0) = p'_1^\mu + p'_2^\mu \quad (12)$$

$$p^\mu p_\mu = (E_1 + E_2)^2 = E^2 = M_0^2 = k^\mu k_\mu$$

- dari kerangka acuan lab. ke kerangka acuan p.m.:

\mathbf{v} = kecepatan pusat massa relatif terhadap laboratorium

$$= \frac{\mathbf{K}}{E_{lab}} = \frac{\mathbf{k}_1}{E_{1,lab} + m_2} = \frac{\mathbf{k}_1}{E_{1k,lab} + m_1 + m_2} \quad (13)$$

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{E_{lab}}{\sqrt{E_{lab}^2 - \mathbf{K}^2}} = \frac{E_{lab}}{M_0}$$

$$E_1 = \gamma(E_{1,lab} - \mathbf{k}_1 \cdot \mathbf{v}) = \frac{E_{lab}}{M_0} \left(E_{1,lab} - \frac{\mathbf{k}_1^2}{E_{lab}} \right) = \frac{1}{M_0} (E_{1,lab}^2 + m_2 E_{1,lab} - \mathbf{k}_1^2)$$

$$= \frac{1}{M_0} (m_1^2 + m_2 E_{1,lab}) \quad (14)$$

$$E_2 = \gamma(E_{2,lab} - \mathbf{k}_2 \cdot \mathbf{v}) = \gamma m_2 = \frac{E_{lab}}{M_0} m_2 = \frac{1}{M_0} (m_2^2 + m_2 E_{1,lab})$$

$$E = E_1 + E_2 = \frac{1}{M_0} (m_1^2 + m_2^2 + 2m_2 E_{1,lab}) = M_0$$

$$\begin{aligned}
\mathbf{p} = \mathbf{p}_1 &= \mathbf{k}_1 + (\gamma - 1)(\mathbf{k}_1 \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} - \gamma E_{1,lab}\mathbf{v} = \mathbf{k}_1 + \left(\frac{E_{lab}}{M_0} - 1 \right) \mathbf{k}_1 - \frac{E_{1,lab}}{M_0} \mathbf{k}_1 \\
&= \frac{m_2}{M_0} \mathbf{k}_1 \\
-\mathbf{p} = \mathbf{p}_2 &= \mathbf{k}_2 + (\gamma - 1)(\mathbf{k}_2 \cdot \hat{\mathbf{v}})\hat{\mathbf{v}} - \gamma E_{2,lab}\mathbf{v} = -\gamma m_2 \mathbf{v} = -\frac{m_2}{M_0} \mathbf{k}_1
\end{aligned} \tag{15}$$

secara umum untuk sembarang waktu dalam proses hamburan (sebut saja keadaan "'):

$$\mathbf{p}'' = \frac{1}{2} (\mathbf{k}_1'' - \mathbf{k}_2'') + \frac{\mathbf{K}}{2M_0} \left(\frac{(\mathbf{k}_1'' - \mathbf{k}_2'') \cdot \mathbf{K}}{E_{lab} + M_0} - (E_{1,lab}'' - E_{2,lab}'') \right) \tag{16}$$

- $E_{1k,lab}$ sebagai input untuk perhitungan dalam kerangka acuan p.m.:

$$\begin{aligned}
E_{lab} &= E_{1,lab} + m_2 = E_{1k,lab} + m_1 + m_2 \\
p = |\mathbf{p}| &= \frac{m_2}{M_0} |\mathbf{k}_1| = \frac{m_2}{M_0} k_1 \\
&= \frac{m_2 \sqrt{E_{1,lab}^2 - m_1^2}}{\sqrt{m_1^2 + m_2^2 + 2m_2 E_{1,lab}}} \\
&= \frac{m_2 \sqrt{(m_1 + E_{1k,lab})^2 - m_1^2}}{\sqrt{m_1^2 + m_2^2 + 2m_2(m_1 + E_{1k,lab})}} \\
&= \frac{m_2 \sqrt{(2m_1 + E_{1k,lab})E_{1k,lab}}}{\sqrt{m_1^2 + m_2^2 + 2m_2(m_1 + E_{1k,lab})}}
\end{aligned} \tag{17}$$

$$\tag{18}$$

- relasi sudut hambur dalam kerangka acuan p.m. θ dan sudut hambur dalam kerangka acuan lab. θ_{lab} :

$$\tan \theta_{lab} = \frac{\sin \theta}{\gamma \left(\cos \theta + v \frac{E_1}{p_1} \right)} = \frac{M_0 \sin \theta}{E_{lab} \left(\cos \theta + \frac{E_1}{E_2} \right)} = \frac{M_0}{E_{lab}} \left(\frac{E_2}{E_1 \sec \theta + E_2} \right) \tan \theta \tag{19}$$

2 Basis states and T-matrix elements (untuk spin total 1/2)

$$|\mathbf{p}\lambda\tau\nu\rangle \equiv |\mathbf{p}\lambda\rangle |\tau\nu\rangle. \tag{20}$$

$$T_{\lambda'\lambda}^{\tau\nu}(\mathbf{p}', \mathbf{p}) \equiv \langle \mathbf{p}'\lambda'\tau\nu | T | \mathbf{p}\lambda\tau\nu \rangle, \tag{21}$$

3 Potential matrix elements (untuk spin total 1/2)

$$\begin{aligned}
V_{\lambda'\lambda}^{\tau\nu}(\mathbf{p}', \mathbf{p}) &= \delta_{\lambda'\lambda} \left[f_0^{\tau\nu}(p', p, \hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) \right. \\
&\quad + \frac{1}{4} f_1^{\tau\nu}(p', p, \hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) \left\{ \cos \theta' \cos \theta + e^{-2i\lambda(\phi' - \phi)} \sin \theta' \sin \theta \right\} \Big] \\
&\quad + \delta_{\lambda', -\lambda} \frac{\lambda}{2} e^{2i\lambda\phi'} f_1^{\tau\nu}(p', p, \hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}) \\
&\quad \times \left\{ \sin \theta' \cos \theta - e^{-2i\lambda(\phi' - \phi)} \cos \theta' \sin \theta \right\}.
\end{aligned} \tag{22}$$

$$V_{\lambda'\lambda}^{\tau\nu}(\mathbf{p}', p\hat{\mathbf{z}}) = e^{-i(\lambda' - \lambda)\phi'} V_{\lambda'\lambda}^{\tau\nu}(p', \theta', p), \tag{23}$$

$$V_{\lambda'\lambda}^{\tau\nu}(\mathbf{p}', p\hat{\mathbf{z}}) = (-)^{\lambda' - \lambda} e^{-2i(\lambda' - \lambda)\phi'} V_{-\lambda', -\lambda}^{\tau\nu}(\mathbf{p}', p\hat{\mathbf{z}}), \tag{24}$$

$$V_{\lambda'\lambda}^{\tau\nu}(p', \theta', p) = (-)^{\lambda' - \lambda} V_{-\lambda', -\lambda}^{\tau\nu}(p', \theta', p). \tag{25}$$

4 T-matrix elements and the Lippmann-Schwinger equation (untuk spin total 1/2)

- relativistic Lippmann-Schwinger (LS) equation for the T-matrix elements:

$$T_{\lambda'\lambda}^{\tau\nu}(\mathbf{p}', \mathbf{p}) = V_{\lambda'\lambda}^{\tau\nu}(\mathbf{p}', \mathbf{p}) + \lim_{\epsilon \rightarrow 0} \sum_{\lambda'' = -\frac{1}{2}}^{\frac{1}{2}} \int d\mathbf{p}'' \frac{V_{\lambda'\lambda''}^{\tau\nu}(\mathbf{p}', \mathbf{p}'')}{z - E_1'' - E_2'' + i\epsilon} T_{\lambda''\lambda}^{\tau\nu}(\mathbf{p}'', \mathbf{p}), \tag{26}$$

dengan $z = E_1 + E_2$, $E_i = \sqrt{m_i^2 + \mathbf{p}_i^2} = \sqrt{m_i^2 + \mathbf{p}^2}$ ($i = 1, 2$ dan $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$).

- rotational invariant, T-matrix elements $T_{\lambda'\lambda}^{\tau\nu}(\mathbf{p}', \mathbf{p})$ bergantung pada $\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}$:

$$T_{\lambda'\lambda}^{\tau\nu}(\mathbf{p}', p\hat{\mathbf{z}}) = e^{-i(\lambda' - \lambda)\phi'} T_{\lambda'\lambda}^{\tau\nu}(p', \theta', p), \tag{27}$$

dengan $T_{\lambda'\lambda}^{\tau\nu}(p', \theta', p)$ mengikuti persamaan:

$$\begin{aligned}
T_{\lambda'\lambda}^{\tau\nu}(p', \theta', p) &= V_{\lambda'\lambda}^{\tau\nu}(p', \theta', p) + \lim_{\epsilon \rightarrow 0} \sum_{\lambda'' = -\frac{1}{2}}^{\frac{1}{2}} \int_0^\infty dp'' \left\{ \frac{p''^2}{z - E_1'' - E_2'' + i\epsilon} \right. \\
&\quad \times \left. \int_{-1}^1 d\cos \theta'' V_{\lambda'\lambda''}^{\tau\nu,\lambda}(p', \theta', p'', \theta'') T_{\lambda''\lambda}^{\tau\nu}(p'', \theta'', p) \right\}.
\end{aligned} \tag{28}$$

dan:

$$V_{\lambda'\lambda''}^{\tau\nu,\lambda}(p', \theta', p'', \theta'') \equiv \int_0^{2\pi} d\phi'' V_{\lambda'\lambda''}^{\tau\nu}(\mathbf{p}', \mathbf{p}'') e^{i(\lambda'\phi' - \lambda''\phi'')} e^{-i\lambda(\phi' - \phi'')} \tag{29}$$

- relasi simetri $T_{\lambda'\lambda}^{\tau\nu}(p', \theta', p)$:

$$T_{\lambda'\lambda}^{\tau\nu}(p', \theta', p) = (-)^{\lambda' - \lambda} T_{-\lambda', -\lambda}^{\tau\nu}(p', \theta', p). \quad (30)$$

Dengan demikian, cukup cari $T_{\frac{1}{2}\frac{1}{2}}^{\tau\nu}(p', \theta', p)$ dan $T_{-\frac{1}{2}\frac{1}{2}}^{\tau\nu}(p', \theta', p)$.

5 Penampang lintang

- S-matrix (amplitudo transisi) dalam kerangka acuan p.m.:

$$S_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p}) = S_{\lambda'\lambda}^\nu(p\hat{\mathbf{p}}', \mathbf{p}) = \delta(\mathbf{p}' - \mathbf{p})\delta_{\lambda'\lambda} - 2\pi i\delta(E' - E)T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p}), \quad (31)$$

dengan

$$T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p}) = T_{\lambda'\lambda}^\nu(p\hat{\mathbf{p}}', \mathbf{p}) = \sum_\tau C^2(\tau_1\tau_2\tau; \nu_1\nu_2\nu) T_{\lambda'\lambda}^{\tau\nu}(p\hat{\mathbf{p}}', \mathbf{p}) \quad (32)$$

Untuk perhitungan penampang lintang hanya diambil suku kedua:

$$S_{\lambda'\lambda}^{\nu(2)}(\mathbf{p}', \mathbf{p}) = -2\pi i\delta(E' - E)T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p}) \quad (33)$$

- transition rate:

$$\begin{aligned} \frac{|S_{\lambda'\lambda}^{\nu(2)}(\mathbf{p}', \mathbf{p})|^2}{\mathcal{T}} &= \frac{2\pi\delta(E' - E)2\pi\delta(E' - E)|T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p})|^2}{\mathcal{T}} \\ &= \frac{2\pi\delta(E' - E)\left(\int_{\mathcal{T}} dt e^{i(E'-E)}\right)|T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p})|^2}{\mathcal{T}} \\ &= \frac{2\pi\delta(E' - E)\left(\int_{\mathcal{T}} dt\right)|T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p})|^2}{\mathcal{T}} \\ &= \frac{2\pi\delta(E' - E)\mathcal{T}|T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p})|^2}{\mathcal{T}} \\ &= 2\pi\delta(E' - E)|T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p})|^2 \end{aligned} \quad (34)$$

- arus partikel terhambur ke keadaan akhir dengan momentum \mathbf{p}' sampai $\mathbf{p}' + d\mathbf{p}'$:

$$d\mathbf{p}' \frac{|S_{\lambda'\lambda}^{\nu(2)}(\mathbf{p}', \mathbf{p})|^2}{\mathcal{T}} = 2\pi dp' p'^2 d\hat{\mathbf{p}}' \delta(E' - E) |T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p})|^2 \quad (35)$$

$$\begin{aligned} E' &= E'_1 + E'_2 = \sqrt{m_1^2 + p'^2} + \sqrt{m_2^2 + p'^2} \\ \frac{dE'}{dp'} &= \frac{p'}{E'_1} + \frac{p'}{E'_2} = \frac{p'(E'_1 + E'_2)}{E'_1 E'_2} = \frac{p'E'}{E'_1 E'_2} \end{aligned} \quad (36)$$

$$\rightarrow d\mathbf{p}' \frac{|S_{\lambda'\lambda}^{\nu(2)}(\mathbf{p}', \mathbf{p})|^2}{\mathcal{T}} = 2\pi dE' \frac{E'_1 E'_2}{E'} p' d\hat{\mathbf{p}}' \delta(E' - E) |T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p})|^2 \quad (37)$$

- arus partikel terhambur ke sudut ruang $d\hat{\mathbf{p}}'$:

$$\begin{aligned}
 dN_{\lambda'\lambda}^\nu &= 2\pi d\hat{\mathbf{p}}' \int dp' p'^2 \delta(E' - E) |T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p})|^2 \\
 &= 2\pi d\hat{\mathbf{p}}' \int dE' \frac{E'_1 E'_2}{E'} p' \delta(E' - E) |T_{\lambda'\lambda}^\nu(\mathbf{p}', \mathbf{p})|^2 \\
 &= 2\pi d\hat{\mathbf{p}}' \frac{E_1 E_2}{E} p |T_{\lambda'\lambda}^\nu(p\hat{\mathbf{p}}', \mathbf{p})|^2
 \end{aligned} \tag{38}$$

(karena $E' = E$, konsekuensinya $p' = p$, $E'_1 = E_1$, $E'_2 = E_2$)

- incoming flux:

$$\begin{aligned}
 \mathbf{j}_0 &= \frac{\mathbf{v}_1 - \mathbf{v}_2}{(2\pi)^3} = \frac{1}{(2\pi)^3} \left(\frac{\mathbf{p}_1}{E_1} - \frac{\mathbf{p}_2}{E_2} \right) = \frac{1}{(2\pi)^3} \left(\frac{\mathbf{p}}{E_1} + \frac{\mathbf{p}}{E_2} \right) = \frac{\mathbf{p}}{(2\pi)^3} \left(\frac{E_1 + E_2}{E_1 E_2} \right) \\
 &= \frac{\mathbf{p}}{(2\pi)^3} \frac{E}{E_1 E_2} \\
 j_0 &= \frac{p}{(2\pi)^3} \frac{E}{E_1 E_2}
 \end{aligned} \tag{39}$$

- penampang lintang:

$$d\sigma_{\lambda'\lambda}^\nu = \frac{dN_{\lambda'\lambda}^\nu}{j_0} = (2\pi)^4 d\hat{\mathbf{p}}' \frac{E_1^2 E_2^2}{E^2} |T_{\lambda'\lambda}^\nu(p\hat{\mathbf{p}}', \mathbf{p})|^2 \tag{40}$$

- spin-averaged differential cross section (untuk spin total 1/2):

$$\begin{aligned}
 \frac{d\sigma}{d\hat{\mathbf{p}}'} &= \frac{1}{2} \sum_{\lambda'\lambda} \frac{d\sigma_{\lambda'\lambda}^\nu}{d\hat{\mathbf{p}}'} = (2\pi)^4 \frac{E_1^2 E_2^2}{E^2} \frac{1}{2} \sum_{\lambda'\lambda} |T_{\lambda'\lambda}^\nu(p\hat{\mathbf{p}}', \mathbf{p})|^2 \\
 &= (2\pi)^4 \frac{E_1^2 E_2^2}{E^2} \frac{1}{2} \sum_{\lambda'\lambda} |T_{\lambda'\lambda}^\nu(p\hat{\mathbf{p}}', \mathbf{p})|^2 \\
 &= (2\pi)^4 \frac{E_1^2 E_2^2}{E^2} \frac{1}{2} \sum_{\lambda'\lambda} |T_{\lambda'\lambda}^\nu(p, \theta', p)|^2 \\
 &= (2\pi)^4 \frac{E_1^2 E_2^2}{E^2} \frac{1}{2} \left(|T_{\frac{1}{2}\frac{1}{2}}^\nu(p, \theta', p)|^2 + |T_{-\frac{1}{2}\frac{1}{2}}^\nu(p, \theta', p)|^2 \right. \\
 &\quad \left. + |T_{\frac{1}{2}, -\frac{1}{2}}^\nu(p, \theta', p)|^2 + |T_{-\frac{1}{2}, -\frac{1}{2}}^\nu(p, \theta', p)|^2 \right) \\
 &= (2\pi)^4 \frac{E_1^2 E_2^2}{E^2} \left(|T_{\frac{1}{2}\frac{1}{2}}^\nu(p, \theta', p)|^2 + |T_{-\frac{1}{2}\frac{1}{2}}^\nu(p, \theta', p)|^2 \right), \tag{41}
 \end{aligned}$$

dengan

$$T_{\lambda'\lambda}^\nu(p, \theta', p) = \sum_\tau C^2(\tau_1 \tau_2 \tau; \nu_1 \nu_2 \nu) T_{\lambda'\lambda}^{\tau\nu}(p, \theta', p) \tag{42}$$

Untuk nonrelativistik $E \rightarrow m$:

$$\frac{d\sigma}{d\hat{\mathbf{p}}'} = (2\pi)^4 \frac{m_1^2 m_2^2}{m^2} \frac{1}{2} \sum_{\lambda'\lambda} |T_{\lambda'\lambda}^\nu(p, \theta', p)|^2$$

$$= (2\pi)^4 \mu^2 \frac{1}{2} \sum_{\lambda' \lambda} |T_{\lambda' \lambda}^\nu(p, \theta', p)|^2 \quad (43)$$