

Numerical realization for relativistic three-dimensional approach based on a simple basis

1 Analytical steps

Note:

$$x = \cos \theta$$

$$\begin{aligned} z &= E_1 + E_2 = \sqrt{m_1^2 + \mathbf{p}^2} + \sqrt{m_2^2 + \mathbf{p}^2} \\ z'' &= E_1'' + E_2'' = \sqrt{m_1^2 + \mathbf{p}''^2} + \sqrt{m_2^2 + \mathbf{p}''^2} \end{aligned}$$

The equation to be solved:

$$\begin{aligned} T_{\lambda'\lambda}(p', p, x') &= V_{\lambda'\lambda}(p', p, x') \\ &+ \lim_{\epsilon \rightarrow 0} \sum_{\lambda''} \int_0^\infty dp'' \frac{p''^2}{z - z'' + i\epsilon} \int_{-1}^1 dx'' V_{\lambda'\lambda''}^\lambda(p', p'', x', x'') T_{\lambda''\lambda}(p'', p, x'') \\ &= V_{\lambda'\lambda}(p', p, x') + \sum_{\lambda''} \int_0^\infty dp'' p''^2 \left\{ \frac{\mathcal{P}}{z - z''} - i\pi\delta(z - z'') \right\} \\ &\quad \times \int_{-1}^1 dx'' V_{\lambda'\lambda''}^\lambda(p', p'', x', x'') T_{\lambda''\lambda}(p'', p, x''). \end{aligned} \quad (1)$$

Take:

$$\begin{aligned} u^2 &= z - (m_1 + m_2), \quad u''^2 = z'' - (m_1 + m_2) \rightarrow z - z'' = u^2 - u''^2 \\ 2u'' du'' &= dz'' = dE_1'' + dE_2'' = \left(\frac{p''}{E_1''} + \frac{p''}{E_2''} \right) dp'' = \frac{z''}{E_1'' E_2''} p'' dp'' \\ \rightarrow p''^2 dp'' &= \frac{E_1'' E_2''}{z''} p'' dz'' = 2 \frac{E_1'' E_2''}{z''} p'' u'' du'' \end{aligned} \quad (2)$$

$$\begin{aligned} T_{\lambda'\lambda}(p', p, x') &= V_{\lambda'\lambda}(p', p, x') + 2 \sum_{\lambda''} \int_{-1}^1 dx'' \int_0^\infty du'' V_{\lambda'\lambda''}^\lambda(p', p'', x', x'') T_{\lambda''\lambda}(p'', p, x'') \frac{E_1'' E_2'' p'' u''}{z''(u^2 - u''^2)} \\ &\quad - i\pi \frac{E_1 E_2}{z} p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \\ &\simeq V_{\lambda'\lambda}(p', p, x') + 2 \sum_{\lambda''} \int_{-1}^1 dx'' \int_0^M du'' V_{\lambda'\lambda''}^\lambda(p', p'', x', x'') T_{\lambda''\lambda}(p'', p, x'') \frac{E_1'' E_2'' p'' u''}{z''(u^2 - u''^2)} \\ &\quad - i\pi \frac{E_1 E_2}{z} p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x''). \end{aligned} \quad (3)$$

Replacing the integral upper limit ∞ with a large number M is acceptable as long as the potential is a short range one. Next, taking care of the singularity.

$$\begin{aligned}
T_{\lambda'\lambda}(p', p, x') &\simeq V_{\lambda'\lambda}(p', p, x') + 2 \sum_{\lambda''} \int_{-1}^1 dx'' \left\{ \int_0^M du'' V_{\lambda'\lambda''}^\lambda(p', p'', x', x'') T_{\lambda''\lambda}(p'', p, x'') \frac{E_1'' E_2'' p'' u''}{z''(u^2 - u''^2)} \right. \\
&\quad \left. - \frac{E_1 E_2}{z} p u V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \int_0^\infty du'' \frac{1}{u^2 - u''^2} \right\} \\
&\quad - i\pi \frac{E_1 E_2}{z} p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \\
&\simeq V_{\lambda'\lambda}(p', p, x') + 2 \sum_{\lambda''} \int_{-1}^1 dx'' \left\{ \int_0^M du'' V_{\lambda'\lambda''}^\lambda(p', p'', x', x'') T_{\lambda''\lambda}(p'', p, x'') \frac{E_1'' E_2'' p'' u''}{z''(u^2 - u''^2)} \right. \\
&\quad \left. - \frac{E_1 E_2}{z} p u V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \int_0^M du'' \frac{1}{u^2 - u''^2} \right. \\
&\quad \left. - \frac{E_1 E_2}{z} p u V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \int_M^\infty du'' \frac{1}{u^2 - u''^2} \right\} \\
&\quad - i\pi \frac{E_1 E_2}{z} p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \\
&\simeq V_{\lambda'\lambda}(p', p, x') + 2 \sum_{\lambda''} \int_{-1}^1 dx'' \int_0^M du'' \frac{1}{u^2 - u''^2} \\
&\quad \times \left\{ \frac{E_1'' E_2''}{z''} p'' u'' V_{\lambda'\lambda''}^\lambda(p', p'', x', x'') T_{\lambda''\lambda}(p'', p, x'') \right. \\
&\quad \left. - \frac{E_1 E_2}{z} p u V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \right\} \\
&\quad - \frac{E_1 E_2}{z} p \ln \left(\frac{M-u}{M+u} \right) \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \\
&\quad - i\pi \frac{E_1 E_2}{z} p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \\
&\simeq V_{\lambda'\lambda}(p', p, x') + 2 \sum_{\lambda''} \int_{-1}^1 dx'' \int_0^M du'' \frac{E_1'' E_2''}{z''} \frac{p'' u''}{u^2 - u''^2} V_{\lambda'\lambda''}^\lambda(p', p'', x', x'') T_{\lambda''\lambda}(p'', p, x'') \\
&\quad - 2 \frac{E_1 E_2}{z} \sum_{\lambda''} \int_{-1}^1 dx'' p u V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'') \int_0^M du'' \frac{1}{u^2 - u''^2} \\
&\quad - \frac{E_1 E_2}{z} p \ln \left(\frac{M-u}{M+u} \right) \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda'\lambda''}^\lambda(p', p, x', x'') T_{\lambda''\lambda}(p, p, x'')
\end{aligned}$$

$$-i\pi \frac{E_1 E_2}{z} p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x''). \quad (4)$$

Let p'' as the integration variable.

$$\begin{aligned} T_{\lambda' \lambda}(p', p, x') &\simeq V_{\lambda' \lambda}(p', p, x') + \sum_{\lambda''} \int_{-1}^1 dx'' \int_0^L dp'' \frac{p''^2}{u^2 - u''^2} V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \\ &\quad - \frac{E_1 E_2}{z} \sum_{\lambda''} \int_{-1}^1 dx'' p u V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \int_0^L dp'' \frac{z''}{E_1'' E_2''} \frac{p''}{u''(u^2 - u''^2)} \\ &\quad - \frac{E_1 E_2}{z} p \ln \left(\frac{M - u}{M + u} \right) \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \\ &\quad - i\pi \frac{E_1 E_2}{z} p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x''), \end{aligned} \quad (5)$$

with

$$M = \sqrt{\sqrt{m_1^2 + L^2} + \sqrt{m_2^2 + L^2} - (m_1 + m_2)} \quad (6)$$

Final equation.

$$\begin{aligned} T_{\lambda' \lambda}(p', p, x') &\simeq V_{\lambda' \lambda}(p', p, x') + \sum_{\lambda''} \int_{-1}^1 dx'' \left[\int_0^L dp'' \frac{p''^2}{u^2 - u''^2} V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \right. \\ &\quad \left. - \frac{E_1 E_2}{z} \left\{ u \int_0^L dp'' \frac{z''}{E_1'' E_2''} \frac{p''}{u''(u^2 - u''^2)} + \ln \left(\frac{M - u}{M + u} \right) + i\pi \right\} \right. \\ &\quad \left. \times p V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \right]. \end{aligned} \quad (7)$$

2 Numerical steps

Discretizing the continuum variables p and x .

$$\begin{array}{lll} p' & \rightarrow & p_i \\ x' & \rightarrow & x_j \end{array} \quad \begin{array}{lll} p'' & \rightarrow & p_k \\ x'' & \rightarrow & x_l \end{array} \quad (8)$$

Changing the integrals into quadratures, setting $p_{n+1} = p$, and using $\bar{\delta}_{ab} = 1 - \delta_{ab}$.

$$\begin{aligned} T_{\lambda' \lambda}(p_i, p, x_j) &= V_{\lambda' \lambda}(p_i, p, x_j) + \sum_{\lambda''} \sum_{l=1}^m w_{x,l} \left[\sum_{k=1}^n w_{p,k} \frac{p_k^2}{u^2 - u_k^2} V_{\lambda' \lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda'' \lambda}(p_k, p, x_l) \right. \\ &\quad \left. - \frac{E_1 E_2}{z} \left\{ u \sum_{r=1}^n \frac{z_r}{E_{r,1} E_{r,2}} \frac{w_{p,r} p_r}{u_r(u^2 - u_r^2)} + \ln \left(\frac{M - u}{M + u} \right) + i\pi \right\} \right. \\ &\quad \left. \times p V_{\lambda' \lambda''}^\lambda(p_i, p, x_j, x_l) T_{\lambda'' \lambda}(p, p, x_l) \right] \end{aligned}$$

$$\begin{aligned}
&= V_{\lambda'\lambda}(p_i, p, x_j) + \sum_{\lambda''}^m \sum_{l=1}^m w_{x,l} \sum_{k=1}^{n+1} \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{u^2 - u_k^2} V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda''\lambda}(p_k, p, x_l) \right. \\
&\quad \left. - \delta_{k,n+1} \frac{E_1 E_2}{z} \left\{ u \sum_{r=1}^n \frac{z_r}{E_{r,1} E_{r,2}} \frac{w_{p,r} p_r}{u_r(u^2 - u_r^2)} + \ln \left(\frac{M-u}{M+u} \right) + i\pi \right\} \right. \\
&\quad \left. \times p V_{\lambda'\lambda''}^\lambda(p_i, p, x_j, x_l) T_{\lambda''\lambda}(p, p, x_l) \right] \\
&= V_{\lambda'\lambda}(p_i, p, x_j) + \sum_{\lambda''}^m \sum_{l=1}^m w_{x,l} \sum_{k=1}^{n+1} \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{u^2 - u_k^2} \right. \\
&\quad \left. - \delta_{k,n+1} \frac{E_1 E_2}{z} p \left\{ u \sum_{r=1}^n \frac{z_r}{E_{r,1} E_{r,2}} \frac{w_{p,r} p_r}{u_r(u^2 - u_r^2)} + \ln \left(\frac{M-u}{M+u} \right) + i\pi \right\} \right] \\
&\quad \times V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda''\lambda}(p_k, p, x_l) . \tag{9}
\end{aligned}$$

Bringing the 'unknowns' to the left side and constructing the system of linear equations.

$$\begin{aligned}
T_{\lambda'\lambda}(p_i, p, x_j) &- \sum_{\lambda''}^m \sum_{l=1}^m w_{x,l} \sum_{k=1}^{n+1} \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{u^2 - u_k^2} \right. \\
&\quad \left. - \delta_{k,n+1} \frac{E_1 E_2}{z} p \left\{ u \sum_{r=1}^n \frac{z_r}{E_{r,1} E_{r,2}} \frac{w_{p,r} p_r}{u_r(u^2 - u_r^2)} + \ln \left(\frac{M-u}{M+u} \right) + i\pi \right\} \right] \\
&\quad \times V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda''\lambda}(p_k, p, x_l) = V_{\lambda'\lambda}(p_i, p, x_j) \\
\sum_{\lambda''}^m \sum_{l=1}^m \sum_{k=1}^{n+1} \left(\delta_{\lambda''\lambda'} \delta_{ki} \delta_{lj} T_{\lambda''\lambda}(p_k, p, x_l) - \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{u^2 - u_k^2} \right. \right. \\
&\quad \left. - \delta_{k,n+1} \frac{E_1 E_2}{z} p \left\{ u \sum_{r=1}^n \frac{z_r}{E_{r,1} E_{r,2}} \frac{w_{p,r} p_r}{u_r(u^2 - u_r^2)} + \ln \left(\frac{M-u}{M+u} \right) + i\pi \right\} \right] \\
&\quad \left. \times w_{x,l} V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda''\lambda}(p_k, p, x_l) \right) = V_{\lambda'\lambda}(p_i, p, x_j) \\
\sum_{\lambda''}^m \sum_{l=1}^m \sum_{k=1}^{n+1} \left(\delta_{\lambda''\lambda'} \delta_{ki} \delta_{lj} - \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{u^2 - u_k^2} \right. \right. \\
&\quad \left. - \delta_{k,n+1} \frac{E_1 E_2}{z} p \left\{ u \sum_{r=1}^n \frac{z_r}{E_{r,1} E_{r,2}} \frac{w_{p,r} p_r}{u_r(u^2 - u_r^2)} + \ln \left(\frac{M-u}{M+u} \right) + i\pi \right\} \right] \\
&\quad \left. \times w_{x,l} V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) \right) T_{\lambda''\lambda}(p_k, p, x_l) = V_{\lambda'\lambda}(p_i, p, x_j) . \tag{10}
\end{aligned}$$

Defining a new indexing scheme.

$$\alpha = (\lambda' - 1)m(n+1) + (j-1)(n+1) + i \quad \text{and} \quad \beta = (\lambda'' - 1)m(n+1) + (l-1)(n+1) + k . \tag{11}$$

Final equation.

$$\sum_{\beta} A_{\alpha\beta}^\lambda(p) T_{\beta,\lambda}(p) = V_{\alpha,\lambda}(p) , \tag{12}$$

with p being the energy parameter and

$$A_{\alpha\beta}^\lambda(p) = \delta_{\lambda''\lambda'} \delta_{ki} \delta_{lj}$$

$$-\left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{u^2 - u_k^2} - \delta_{k,n+1} \frac{E_1 E_2}{z} p \left\{ u \sum_{r=1}^n \frac{z_r}{E_{r,1} E_{r,2}} \frac{w_{p,r} p_r}{u_r(u^2 - u_r^2)} + \ln \left(\frac{M-u}{M+u} \right) + i\pi \right\} \right] \\ \times w_{x,l} V_{\lambda' \lambda''}^{\lambda}(p_i, p_k, x_j, x_l) \quad (13)$$

$$T_{\beta,\lambda}(p) = T_{\lambda''\lambda}(p_k, p, x_l) \quad (14)$$

$$V_{\alpha,\lambda}(p) = V_{\lambda'\lambda}(p_i, p, x_j). \quad (15)$$