

Numerical realization for three-dimensional approach

1 Analytical steps

The equation to be solved, note that $x = \cos \theta$:

$$\begin{aligned}
T_{\lambda' \lambda}(p', p, x') &= V_{\lambda' \lambda}(p', p, x') \\
&\quad + \lim_{\epsilon \rightarrow 0} \sum_{\lambda''} \int_0^\infty dp'' \frac{p''^2}{\frac{p^2}{2\mu} + i\epsilon - \frac{p''^2}{2\mu}} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \\
&= V_{\lambda' \lambda}(p', p, x') \\
&\quad + 2\mu \lim_{\epsilon \rightarrow 0} \sum_{\lambda''} \int_0^\infty dp'' \frac{p''^2}{p^2 + i\epsilon - p''^2} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \\
&= V_{\lambda' \lambda}(p', p, x') + 2\mu \sum_{\lambda''} \int_0^\infty dp'' p''^2 \left\{ \frac{\mathcal{P}}{p^2 - p''^2} - i\pi\delta(p^2 - p''^2) \right\} \\
&\quad \times \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \\
&= V_{\lambda' \lambda}(p', p, x') + 2\mu \sum_{\lambda''} \int_{-1}^1 dx'' \int_0^\infty dp'' V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \frac{p''^2}{p^2 - p''^2} \\
&\quad - i\pi\mu p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \\
&\simeq V_{\lambda' \lambda}(p', p, x') + 2\mu \sum_{\lambda''} \int_{-1}^1 dx'' \int_0^M dp'' V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \frac{p''^2}{p^2 - p''^2} \\
&\quad - i\pi\mu p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x''). \tag{1}
\end{aligned}$$

Replacing the integral upper limit ∞ with a large number M is acceptable as long as the potential is a short range one. Next, taking care of the singularity.

$$\begin{aligned}
T_{\lambda' \lambda}(p', p, x') &\simeq V_{\lambda' \lambda}(p', p, x') + 2\mu \sum_{\lambda''} \int_{-1}^1 dx'' \left\{ \int_0^M dp'' p''^2 V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \frac{1}{p^2 - p''^2} \right. \\
&\quad \left. - p^2 V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \int_0^\infty dp'' \frac{1}{p^2 - p''^2} \right\} \\
&\quad - i\pi\mu p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^{\pi\lambda}(p', p, x', x'') T_{\lambda'' \lambda}^\pi(p, p, x'')
\end{aligned}$$

$$\begin{aligned}
& \simeq V_{\lambda' \lambda}(p', p, x') + 2\mu \sum_{\lambda''} \int_{-1}^1 dx'' \left\{ \int_0^M dp'' p''^2 V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \frac{1}{p^2 - p''^2} \right. \\
& \quad \left. - p^2 V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \int_0^M dp'' \frac{1}{p^2 - p''^2} \right. \\
& \quad \left. - p^2 V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \int_M^\infty dp'' \frac{1}{p^2 - p''^2} \right\} \\
& \quad - i\pi\mu p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \\
& \simeq V_{\lambda' \lambda}(p', p, x') + 2\mu \sum_{\lambda''} \int_{-1}^1 dx'' \int_0^M dp'' \frac{1}{p^2 - p''^2} \\
& \quad \times \left\{ p''^2 V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') - p^2 V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \right\} \\
& \quad - \mu p \ln \left(\frac{M-p}{M+p} \right) \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \\
& \quad - i\pi\mu p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \\
& \simeq V_{\lambda' \lambda}(p', p, x') + 2\mu \sum_{\lambda''} \int_{-1}^1 dx'' \int_0^M dp'' \frac{p''^2}{p^2 - p''^2} V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \\
& \quad - 2\mu \sum_{\lambda''} \int_{-1}^1 dx'' p^2 V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \int_0^M dp'' \frac{1}{p^2 - p''^2} \\
& \quad - \mu p \ln \left(\frac{M-p}{M+p} \right) \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \\
& \quad - i\pi\mu p \sum_{\lambda''} \int_{-1}^1 dx'' V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x''). \tag{2}
\end{aligned}$$

Final equation.

$$\begin{aligned}
T_{\lambda' \lambda}(p', p, x') & \simeq V_{\lambda' \lambda}(p', p, x') + 2\mu \sum_{\lambda''} \int_{-1}^1 dx'' \left[\int_0^M dp'' \frac{p''^2}{p^2 - p''^2} V_{\lambda' \lambda''}^\lambda(p', p'', x', x'') T_{\lambda'' \lambda}(p'', p, x'') \right. \\
& \quad \left. - \left\{ p \int_0^M dp'' \frac{1}{p^2 - p''^2} + \frac{1}{2} \ln \left(\frac{M-p}{M+p} \right) + \frac{1}{2} i\pi \right\} \right. \\
& \quad \left. \times p V_{\lambda' \lambda''}^\lambda(p', p, x', x'') T_{\lambda'' \lambda}(p, p, x'') \right]. \tag{3}
\end{aligned}$$

2 Numerical steps

Discretizing the continuum variables p and x .

$$\begin{aligned} p' &\rightarrow p_i & p'' &\rightarrow p_k \\ x' &\rightarrow x_j & x'' &\rightarrow x_l \end{aligned} \quad (4)$$

Changing the integrals into quadratures, setting $p_{n+1} = p$, and using $\bar{\delta}_{ab} = 1 - \delta_{ab}$.

$$\begin{aligned} T_{\lambda'\lambda}(p_i, p, x_j) &= V_{\lambda'\lambda}(p_i, p, x_j) + 2\mu \sum_{\lambda''} \sum_{l=1}^m w_{x,l} \left[\sum_{k=1}^n w_{p,k} \frac{p_k^2}{p^2 - p_k^2} V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda''\lambda}(p_k, p, x_l) \right. \\ &\quad \left. - \left\{ p \sum_{r=1}^n w_{p,r} \frac{1}{p^2 - p_r^2} + \frac{1}{2} \ln \left(\frac{M-p}{M+p} \right) + \frac{1}{2} i\pi \right\} p V_{\lambda'\lambda''}^\lambda(p_i, p, x_j, x_l) T_{\lambda''\lambda}(p, p, x_l) \right] \\ &= V_{\lambda'\lambda}(p_i, p, x_j) + 2\mu \sum_{\lambda''} \sum_{l=1}^m w_{x,l} \sum_{k=1}^{n+1} \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{p^2 - p_k^2} V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda''\lambda}(p_k, p, x_l) \right. \\ &\quad \left. - \delta_{k,n+1} \left\{ p \sum_{r=1}^n \frac{w_{p,r}}{p^2 - p_r^2} + \frac{1}{2} \ln \left(\frac{M-p}{M+p} \right) + \frac{1}{2} i\pi \right\} p V_{\lambda'\lambda''}^\lambda(p_i, p, x_j, x_l) T_{\lambda''\lambda}(p, p, x_l) \right] \\ &= V_{\lambda'\lambda}(p_i, p, x_j) + 2\mu \sum_{\lambda''} \sum_{l=1}^m w_{x,l} \\ &\quad \times \sum_{k=1}^{n+1} \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{p^2 - p_k^2} - \delta_{k,n+1} p \left\{ p \sum_{r=1}^n \frac{w_{p,r}}{p^2 - p_r^2} + \frac{1}{2} \ln \left(\frac{M-p}{M+p} \right) + \frac{1}{2} i\pi \right\} \right] \\ &\quad \times V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda''\lambda}(p_k, p, x_l). \end{aligned} \quad (5)$$

Bringing the 'unknowns' to the left side and constructing the system of linear equations.

$$\begin{aligned} T_{\lambda'\lambda}(p_i, p, x_j) - 2\mu \sum_{\lambda''} \sum_{l=1}^m w_{x,l} \\ \sum_{k=1}^{n+1} \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{p^2 - p_k^2} - \delta_{k,n+1} p \left\{ p \sum_{r=1}^n \frac{w_{p,r}}{p^2 - p_r^2} + \frac{1}{2} \ln \left(\frac{M-p}{M+p} \right) + \frac{1}{2} i\pi \right\} \right] \\ \times V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda''\lambda}(p_k, p, x_l) = V_{\lambda'\lambda}(p_i, p, x_j) \\ \sum_{\lambda''} \sum_{l=1}^m \sum_{k=1}^{n+1} \left(\delta_{\lambda''\lambda'} \delta_{ki} \delta_{lj} T_{\lambda''\lambda}(p_k, p, x_l) \right. \\ \left. - \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{p^2 - p_k^2} - \delta_{k,n+1} p \left\{ p \sum_{r=1}^n \frac{w_{p,r}}{p^2 - p_r^2} + \frac{1}{2} \ln \left(\frac{M-p}{M+p} \right) + \frac{1}{2} i\pi \right\} \right] \right. \\ \left. \times 2\mu w_{x,l} V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) T_{\lambda''\lambda}(p_k, p, x_l) \right) = V_{\lambda'\lambda}(p_i, p, x_j) \\ \sum_{\lambda''} \sum_{l=1}^m \sum_{k=1}^{n+1} \left(\delta_{\lambda''\lambda'} \delta_{ki} \delta_{lj} \right. \\ \left. - \left[\bar{\delta}_{k,n+1} \frac{w_{p,k} p_k^2}{p^2 - p_k^2} - \delta_{k,n+1} p \left\{ p \sum_{r=1}^n \frac{w_{p,r}}{p^2 - p_r^2} + \frac{1}{2} \ln \left(\frac{M-p}{M+p} \right) + \frac{1}{2} i\pi \right\} \right] \right. \\ \left. \times 2\mu w_{x,l} V_{\lambda'\lambda''}^\lambda(p_i, p_k, x_j, x_l) \right) T_{\lambda''\lambda}(p_k, p, x_l) = V_{\lambda'\lambda}(p_i, p, x_j). \end{aligned} \quad (6)$$

Defining a new indexing scheme.

$$\alpha = (\lambda' - 1)m(n+1) + (j-1)(n+1) + i \quad \text{and} \quad \beta = (\lambda'' - 1)m(n+1) + (l-1)(n+1) + k. \quad (7)$$

Final equation.

$$\sum_{\beta} A_{\alpha\beta}^{\lambda}(p) T_{\beta,\lambda}(p) = V_{\alpha,\lambda}(p), \quad (8)$$

with p being the energy parameter and

$$A_{\alpha\beta}^{\lambda}(p) = \delta_{\lambda''\lambda'}\delta_{ki}\delta_{lj} - \left[\bar{\delta}_{k,n+1} \frac{w_{p,k}p_k^2}{p^2 - p_k^2} - \delta_{k,n+1}p \left\{ p \sum_{r=1}^n \frac{w_{p,r}}{p^2 - p_r^2} + \frac{1}{2} \ln \left(\frac{M-p}{M+p} \right) + \frac{1}{2}i\pi \right\} \right] \\ \times 2\mu w_{x,l} V_{\lambda'\lambda''}^{\lambda}(p_i, p_k, x_j, x_l) \quad (9)$$

$$T_{\beta,\lambda}(p) = T_{\lambda''\lambda}(p_k, p, x_l) \quad (10)$$

$$V_{\alpha,\lambda}(p) = V_{\lambda'\lambda}(p_i, p, x_j). \quad (11)$$