Metode Numerik

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Daftar Pustaka:

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Isi

Solusi Sistem Persamaan Linear

Sistem persamaan linear:

$$
\begin{array}{cccccc}
a_{11}x_1 & +a_{12}x_2 & +a_{13}x_3 & \cdots & +a_{1n}x_n & =b_1 \\
a_{21}x_1 & +a_{22}x_2 & +a_{23}x_3 & \cdots & +a_{2n}x_n & =b_2 \\
a_{31}x_1 & +a_{32}x_2 & +a_{33}x_3 & \cdots & +a_{3n}x_n & =b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1}x_1 & +a_{n2}x_2 & +a_{n3}x_3 & \cdots & +a_{nn}x_n & =b_n
\end{array}
$$

n buah persamaan dengan n buah unknown x_j

a_{ij} dan b_i diketahui

i, j = 1, 2, …, n

$x_j = ?$

Soal:

$$2x-3y+2z=-6 \quad (1)$$
$$-x+2y-3z=2 \quad (2)$$
$$x+y-z=0 \quad (3)$$

3 persamaan dan
3 unknown

Jawab:

$$2x-3y+2z=-6 \quad (1)$$
$$0.5y-2z=-1 \quad (2)$$
$$2.5y-2z=3 \quad (3)$$

eliminasi x:
pers. (2) + 0.5 pers. (1)
pers. (3) – 0.5 pers. (1)

$$2x-3y+2z=-6 \quad (1)$$
$$0.5y-2z=-1 \quad (2)$$
$$8z=8 \quad (3)$$

eliminasi y:
pers. (3) – 5 pers. (2)

$$z=1$$
$$y=\frac{-1+2z}{0.5}=2$$
$$x=\frac{-6+3y-2z}{2}=-1$$

substitusi mundur:
pers. (3) → mencari z
pers. (2) → mencari y
pers. (1) → mencari x

Dalam bentuk matriks:

Soal:

$$\begin{pmatrix} 2 & -3 & 2 \\ -1 & 2 & -3 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix}$$

Jawab:

$$\begin{pmatrix} 2 & -3 & 2 \\ 0 & 0.5 & -2 \\ 0 & 2.5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -3 & 2 \\ 0 & 0.5 & -2 \\ 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ -1 \\ 8 \end{pmatrix}$$

$$z = 1$$

$$y = \frac{-1 + 2z}{0.5} = 2$$

$$x = \frac{-6 + 3y - 2z}{2} = -1$$

Eliminasi Gauss

Metode Eliminasi Gauss mencari solusi sebuah sistem persamaan linear dengan cara seperti ditunjukkan pada contoh sebelum ini:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

$$a_{ij}^{(0)} = a_{ij}, \quad b_i^{(0)} = b_i$$

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)}$$

$$b_i^{(k)} = b_i^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} b_k^{(k-1)}$$

$$(k = 1, \ldots, n-1;$$
$$i = k+1, \ldots, n; \; j = k, \ldots, n)$$

$a_{ij}^{(m)}, b_i^{(m)} \equiv a_{ij}, b_i$ pada langkah ke m

$$\begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \cdots & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(n-1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1^{(0)} \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(n-1)} \end{pmatrix}$$

→ halaman berikut

Substitusi mundur:

$$\begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & \cdots & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ 0 & 0 & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^{(n-1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1^{(0)} \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(n-1)} \end{pmatrix}$$

$$\downarrow$$

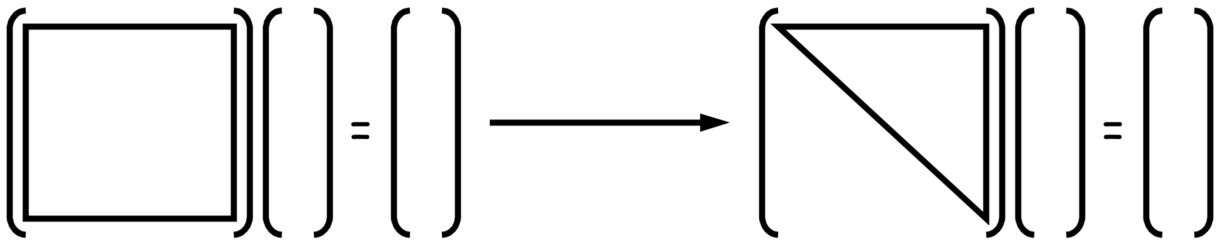
$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_{n-j} = \frac{b_{n-j}^{(n-j-1)} - \sum_{k=n-j+1}^{n} a_{n-j,k}^{(n-j-1)} x_k}{a_{n-j,n-j}^{(n-j-1)}} \qquad (j = 1, \ldots, n-1)$$

$$\begin{pmatrix} & A & \end{pmatrix} \begin{pmatrix} X \end{pmatrix} = \begin{pmatrix} B \end{pmatrix} \quad \text{atau} \quad AX = B$$

Jadi, metode Eliminasi Gauss terdiri dari dua tahap:

1. triangulasi: mengubah matriks A menjadi matriks segitiga (matriks B dengan begitu juga berubah)

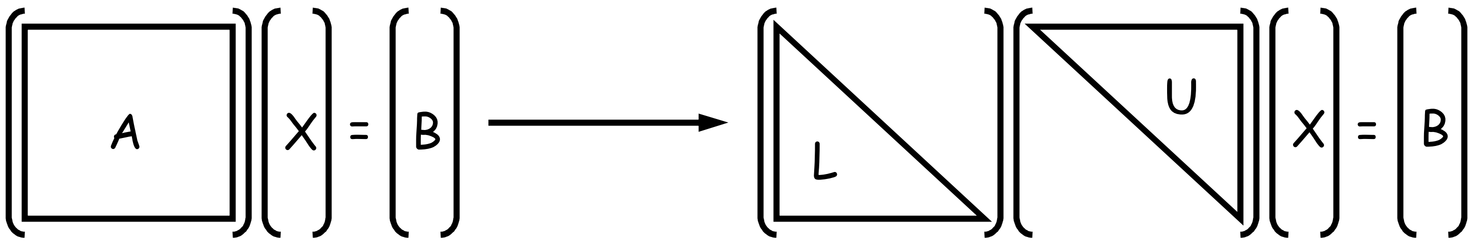


2. substitusi mundur: menghitung x mengikuti urutan terbalik, dari yang terakhir (x_n) sampai yang pertama (x_1)

LU Decomposition

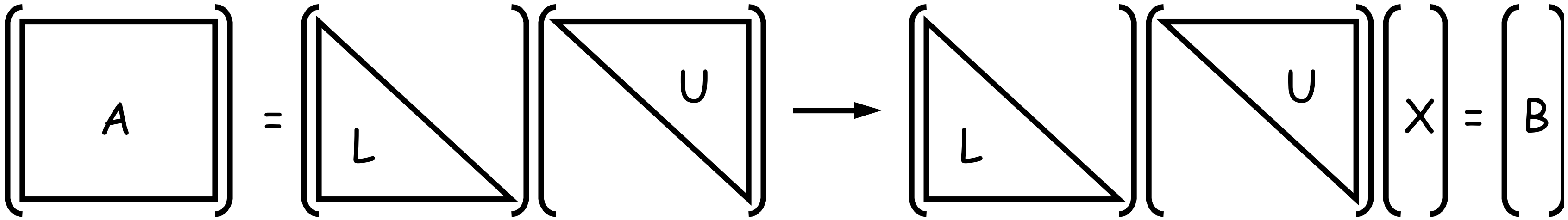
$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} B \end{bmatrix} \quad \text{atau} \quad AX = B$$

Pada metode LU Decomposition, matriks A ditulis ulang sebagai perkalian matriks L dan U (matriks A diurai menjadi matriks L dan U). Matriks L dan U merupakan matriks segitiga. Matriks B tidak berubah, karena matriks A tidak berubah, melainkan hanya ditulis ulang.

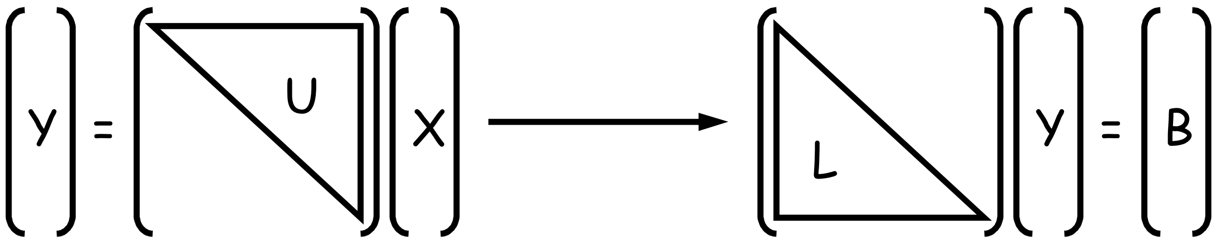


Langkah:

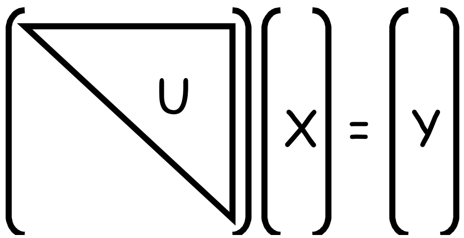
1. Cari matriks L dan U sehingga A = LU. Matriks B tetap.



2. Definisikan sebuah matriks kolom baru, misalnya Y, yaitu Y = UX, sehingga LY = B. Lalu hitung y dengan substitusi maju (mulai dari y_1 sampai y_n).



3. Hitung x dengan substitusi mundur (mulai dari x_n sampai x_1).



Jelas bahwa metode LU Decomposition pada prinsipnya sama dengan metode Eliminasi Gauss: matriks U merupakan hasil triangulasi matriks A, yang juga mengakibatkan B berubah menjadi Y.

Mencari matriks L dan U:

$$\begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

Diperoleh:

$$a_{i1} = l_{i1} \quad \rightarrow \quad l_{i1} = a_{i1} \quad (i = 1,\dots,n)$$

$$a_{1j} = l_{11}u_{1j} \quad \rightarrow \quad u_{1j} = \frac{a_{1j}}{l_{11}} \quad (j = 2,\dots,n)$$

$$a_{i2} = l_{i1}u_{12} + l_{i2} \quad \rightarrow \quad l_{i2} = a_{i2} - l_{i1}u_{12} \quad (i = 2,\dots,n)$$

$$a_{2j} = l_{21}u_{1j} + l_{22}u_{2j} \quad \rightarrow \quad u_{2j} = \frac{a_{2j} - l_{21}u_{1j}}{l_{22}} \quad (j = 3,\dots,n)$$

$$a_{i3} = l_{i1}u_{13} + l_{i2}u_{23} + l_{i3} \quad \rightarrow \quad l_{i3} = a_{i3} - l_{i1}u_{13} - l_{i2}u_{23} \quad (i = 3,\dots,n)$$

$$a_{3j} = l_{31}u_{1j} + l_{32}u_{2j} + l_{33}u_{3j} \quad \rightarrow \quad u_{3j} = \frac{a_{3j} - l_{31}u_{1j} - l_{32}u_{2j}}{l_{33}} \quad (j = 4,\dots,n)$$

$\dots$

Jadi, elemen matriks L dan U dicari menurut:

$$l_{i1} = a_{i1} \qquad (i = 1, \ldots, n)$$

$$u_{1j} = \frac{a_{1j}}{l_{11}} \qquad (j = 2, \ldots, n)$$

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \qquad (i = j, \ldots, n;\ j = 2, \ldots, n)$$

$$u_{ij} = \frac{a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}}{l_{ii}} \qquad (j = i+1, \ldots, n;\ i = 2, \ldots, n-1)$$

secara bergantian:

1. matriks L kolom 1, matriks U baris 1
2. matriks L kolom 2, matriks U baris 2
3. …
4. matriks L kolom (n-1), matriks U baris (n-1)
5. matriks L kolom n

Substitusi maju untuk menghitung y:

$$\begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

$$y_1 = \frac{b_1}{l_{11}}$$

$$y_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} y_j}{l_{ii}} \quad (i = 2, \ldots, n)$$

Substitusi mundur untuk menghitung x:

$$\begin{pmatrix} 1 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 1 & u_{23} & \cdots & u_{2n} \\ 0 & 0 & 1 & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{pmatrix}$$

$$x_n = y_n$$

$$x_{n-i} = y_{n-i} - \sum_{j=n-i+1}^{n} u_{n-i,j} x_j \quad (i = 1, \ldots, n-1)$$

Kembali ke soal $AX = B$, dengan $A = \begin{pmatrix} 2 & -3 & 2 \\ -1 & 2 & -3 \\ 1 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} -6 \\ 2 \\ 0 \end{pmatrix}$.

Jawab:

$$A = LU \longrightarrow L = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0.5 & 0 \\ 1 & 2.5 & 8 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1.5 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Y = UX, \quad LY = B \longrightarrow Y = \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix}$$

$$UX = Y \longrightarrow X = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$