

# Jacobi Coordinates

## A. Two Particles

masses:

$$m_1 \ m_2$$

**position, momentum and energy in lab. frame**

of the two particles:

$$m_1, \vec{x}_1, \vec{p}_1 \quad m_2, \vec{x}_2, \vec{p}_2$$

$$\vec{p}_1 = m_1 \dot{\vec{x}}_1 \quad \vec{p}_2 = m_2 \dot{\vec{x}}_2$$

$$E_{k,1}^{lab} = \frac{p_1^2}{2m_1} \quad E_{k,2}^{lab} = \frac{p_2^2}{2m_2}$$

$$\begin{aligned} E_k^{lab} &= E_{k,1}^{lab} + E_{k,2}^{lab} \\ &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \end{aligned}$$

of the center of mass/momentum:

$$m, \vec{x}, \vec{p}$$

$$m = m_1 + m_2$$

$$\vec{x} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m}$$

$$\begin{aligned} \vec{p} &= m \dot{\vec{x}} \\ &= \vec{p}_1 + \vec{p}_2 \end{aligned}$$

$$\begin{aligned} E_{k_{cm}}^{lab} &= \frac{p^2}{2m} \\ &= \frac{(\vec{p}_1 + \vec{p}_2)^2}{2(m_1 + m_2)} \end{aligned}$$

## position, momentum and energy in c.m. frame

of the two particles:

$$m_1, \vec{\xi}_1, \vec{\eta}_1 \quad m_2, \vec{\xi}_2, \vec{\eta}_2$$

$$\vec{\xi}_1 = \vec{x}_1 - \vec{x} \quad \vec{\xi}_2 = \vec{x}_2 - \vec{x}$$

$$\vec{\eta}_1 = m_1 \dot{\vec{\xi}}_1 \quad \vec{\eta}_2 = m_2 \dot{\vec{\xi}}_2$$

$$E_{k,1}^{cm} = \frac{\eta_1^2}{2m_1} \quad E_{k,2}^{cm} = \frac{\eta_2^2}{2m_2}$$

$$\begin{aligned} E_k^{cm} &= E_{k,1}^{cm} + E_{k,2}^{cm} \\ &= \frac{\eta_1^2}{2m_1} + \frac{\eta_2^2}{2m_2} \end{aligned}$$

of the center of mass/momentum:

$$m, \vec{\xi}, \vec{\eta}$$

$$\begin{aligned} \vec{\xi} &= \frac{m_1 \vec{\xi}_1 + m_2 \vec{\xi}_2}{m} \\ &= \frac{m_1(\vec{x}_1 - \vec{x}) + m_2(\vec{x}_2 - \vec{x})}{m} \\ &= \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2 + (m_1 + m_2) \vec{x}}{m} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \vec{\eta} &= \vec{\eta}_1 + \vec{\eta}_2 \\ &= m_1 \dot{\vec{\xi}}_1 + m_2 \dot{\vec{\xi}}_2 \\ &= m \dot{\vec{\xi}} \\ &= 0 \end{aligned}$$

$$\begin{aligned} E_{k_{cm}}^{cm} &= \frac{\eta^2}{2m} \\ &= 0 \end{aligned}$$

**relation between position, momentum and energy in lab - cm frames**

$$\vec{x}_1 = \vec{x} + \vec{\xi}_1 \quad \vec{x}_2 = \vec{x} + \vec{\xi}_2$$

$$\begin{aligned}\vec{p}_1 &= m_1 \dot{\vec{x}}_1 \\ &= m_1 (\dot{\vec{x}} + \dot{\vec{\xi}}_1) \\ &= \frac{m_1}{m} \vec{p} + \vec{\eta}_1 \\ \vec{p}_2 &= m_2 \dot{\vec{x}}_2 \\ &= m_2 (\dot{\vec{x}} + \dot{\vec{\xi}}_2) \\ &= \frac{m_2}{m} \vec{p} + \vec{\eta}_2\end{aligned}$$

$$E_k^{lab} = E_{k_{cm}}^{lab} + E_k^{cm}$$

Proof for the last equation:

$$\begin{aligned}(1) \quad E_k^{lab} &= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \\ &= \frac{1}{2} m_1 (\dot{\vec{x}} + \dot{\vec{\xi}}_1)^2 + \frac{1}{2} m_2 (\dot{\vec{x}} + \dot{\vec{\xi}}_2)^2 \\ &= \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} (m_1 \dot{\xi}_1^2 + m_2 \dot{\xi}_2^2) + \dot{\vec{x}} \cdot (m_1 \dot{\xi}_1 + m_2 \dot{\xi}_2) \\ &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m_1 \dot{\xi}_1^2 + \frac{1}{2} m_2 \dot{\xi}_2^2 \\ &= E_{k_{cm}}^{lab} + E_k^{cm}\end{aligned}$$

$$\begin{aligned}(2) \quad E_k^{lab} &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \\ &= \frac{1}{2m_1} \left( \frac{m_1}{m} \vec{p} + \vec{\eta}_1 \right)^2 + \frac{1}{2m_2} \left( \frac{m_2}{m} \vec{p} + \vec{\eta}_2 \right)^2 \\ &= \frac{p^2}{2m^2} (m_1 + m_2) + \frac{\eta_1^2}{2m_1} + \frac{\eta_2^2}{2m_2} + \frac{\vec{p}}{m} \cdot (\vec{\eta}_1 + \vec{\eta}_2) \\ &= \frac{p^2}{2m} + \frac{\eta_1^2}{2m_1} + \frac{\eta_2^2}{2m_2} \\ &= E_{k_{cm}}^{lab} + E_k^{cm}\end{aligned}$$

### Jacobi coordinates:

$$\vec{r} = \vec{x}_2 - \vec{x}_1 \quad \text{relative position particle 2 to particle 1}$$

$$\begin{aligned}\mu &= \frac{m_1 m_2}{m_1 + m_2} \\ &= \frac{m_1 m_2}{m} \quad \text{reduced mass}\end{aligned}$$

$$\begin{aligned}\vec{q} &= \mu \dot{\vec{r}} \\ &= \frac{m_1 m_2}{m} (\dot{\vec{x}}_2 - \dot{\vec{x}}_1) \\ &= \frac{1}{m} (m_1 \vec{p}_2 - m_2 \vec{p}_1) \quad \text{relative momentum}\end{aligned}$$

$$\begin{aligned}E_k &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} \\ &= \frac{p^2}{2m} + \frac{q^2}{2\mu} \\ E_k &= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \\ &= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \mu \dot{r}^2\end{aligned}$$

Proof:

$$\begin{aligned}(1) \quad E_k &= \frac{p^2}{2m} + \frac{q^2}{2\mu} \\ &= \frac{(p_1 + p_2)^2}{2m} + \frac{(m_1 \vec{p}_2 - m_2 \vec{p}_1)^2}{2m_1 m_2 m} \\ &= \frac{(p_1^2 + p_2^2)}{2m} + \frac{1}{2m} \left( \frac{m_1}{m_2} p_2^2 + \frac{m_2}{m_1} p_1^2 \right) \\ &= \frac{p_1^2}{2m} \left( 1 + \frac{m_2}{m_1} \right) + \frac{p_2^2}{2m} \left( 1 + \frac{m_1}{m_2} \right) \\ &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}\end{aligned}$$

$$\begin{aligned}(2) \quad \vec{\xi}_1 &= \vec{x}_1 - \vec{x} \\ &= \vec{x}_1 - \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m} \\ &= \frac{m_2}{m} (\vec{x}_1 - \vec{x}_2) \\ &= -\frac{m_2}{m} \vec{r}\end{aligned}$$

$$\begin{aligned}
\vec{\xi}_2 &= \frac{m_1}{m} \vec{r} \\
E_k &= \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \\
&= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m_1 \dot{\xi}_1^2 + \frac{1}{2} m_2 \dot{\xi}_2^2 \\
&= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m_1 \frac{m_2^2}{m^2} \dot{r}^2 + \frac{1}{2} m_2 \frac{m_1^2}{m^2} \dot{r}^2 \\
&= \frac{1}{2} m \dot{x}^2 + \frac{\dot{r}^2}{2m^2} m_1 m_2 (m_1 + m_2) \\
&= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \mu \dot{r}^2
\end{aligned}$$

Note: With Jacobi coordinates one can work easily in lab. frame and in c.m. frame. The Jacobi coordinates have been expressed in notation of lab. frame. But one may take it as general expression (that is why the equation is not labeled with 'lab'). One may want to express the Jacobi coordinates with notation of c.m. frame when one is working in c.m. frame, but it does not make any change. The form of equations will essentially remain the same; for instance the kinetical energy

$$E_k^{cm} = \frac{\eta^2}{2m} + \frac{q^2}{2\mu}$$

(The first term is the kinetical energy of the center of mass in center of mass frame which is actually zero.) It is obvious that the form essentially remains the same.

## B. Three Particles

**masses**

three particles:

$$m_1, m_2, m_3$$

or two particles:

$$m, m_3 \quad (m = m_1 + m_2)$$

**reduced mass**

for [(1) - (2)] system:

$$\begin{aligned} \mu &= \frac{m_1 m_2}{m_1 + m_2} \\ &= \frac{m_1 m_2}{m} \end{aligned}$$

for [(3) - (12)] system:

$$\begin{aligned} \tilde{\mu} &= \frac{m m_3}{m + m_3} \\ &= \frac{m m_3}{M} \end{aligned}$$

$$M = m_1 + m_2 + m_3$$

**position and momentum**

of the center of mass/momentum:

$$\begin{aligned} \vec{Y} &= \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2 + m_3 \vec{x}_3}{M} \\ &= \frac{m \vec{x} + m_3 \vec{x}_3}{M} \end{aligned}$$

$$\begin{aligned} \vec{K} &= M \dot{\vec{Y}} \\ &= m_1 \dot{\vec{x}}_1 + m_2 \dot{\vec{x}}_2 + m_3 \dot{\vec{x}}_3 \\ &= m \dot{\vec{x}} + m_3 \dot{\vec{x}}_3 \\ &= \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \\ &= \vec{p} + \vec{p}_3 \end{aligned}$$

## jacobi coordinates

position:

for [(1) - (2)] system:

$$\vec{r} = \vec{x}_2 - \vec{x}_1$$

for [(3) - (12)] system:

$$\begin{aligned}\vec{R} &= \vec{x}_3 - \vec{x} \\ &= \vec{x}_3 - \frac{1}{m}(m_1\vec{x}_1 + m_2\vec{x}_2)\end{aligned}$$

momentum:

for [(1) - (2)] system:

$$\vec{q} = \frac{1}{m}(m_1\vec{p}_2 - m_2\vec{p}_1)$$

for [(3) - (12)] system:

$$\begin{aligned}\vec{Q} &= \frac{1}{M}(m\vec{p}_3 - m_3\vec{p}) \\ &= \frac{1}{m_1 + m_2 + m_3}((m_1 + m_2)\vec{p}_3 - m_3(\vec{p}_1 + \vec{p}_2))\end{aligned}$$

kinetical energy:

$$\begin{aligned}E_k &= \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} + \frac{\vec{p}_3^2}{2m_3} \\ &= \frac{\vec{p}^2}{2m} + \frac{q^2}{2\mu} + \frac{\vec{p}_3^2}{2m_3} \\ &= \frac{K^2}{2M} + \frac{q^2}{2\mu} + \frac{Q^2}{2\tilde{\mu}}\end{aligned}$$

## C. More Particles

For more particles the Jacobi coordinates can be derived easily based on two-particle system.