Hamburan Dua Partikel Identik

Perhatikan video di https://youtu.be/2LxRC3J-Hd4, yang menggambarkan hamburan partikel identik dalam kerangka acuan pusat massa.

- Apabila proyektil dan target partikel-partikel yang identik, kita tidak tahu, apakah yang kita deteksi di keadaan akhir adalah proyektil terhambur atau target terpental. Dalam kerangka acuan pusat massa seperti digambarkan dalam video tersebut, kita tidak tahu apakah proses A atau proses B yang terjadi. Tidak dapat dipastikan apakah partikel-partikel yang dideteksi semuanya sampai ke detektor melalui proses A saja atau proses B saja. Kedua proses itu terjadi dan, karena itu, kedua-duanya proses A dan proses B harus diperhitungkan.

- Dalam mekanika kuantum, tiap proses A dan B dinyatakan oleh fungsi gelombang. Memperhitungkan keduanya berarti menjumlahkan fungsi gelombang untuk proses A dan fungsi gelombang untuk proses B. Jadi, fungsi gelombang hamburan merupakan superposisi fungsi gelombang proses A dan fungsi gelombang proses B.

- Lihat lagi fungsi gelombang hamburan:

$$\psi_p^{(+)}(\mathbf{r}) = \varphi_p(\mathbf{r}) + A(\mathbf{p}', \mathbf{p}) \frac{1}{(2\pi)^{3/2}} \frac{e^{ipr}}{r} \,. \tag{1}$$

  Perhatikan suku kedua fungsi gelombang hamburan di Eq. (1), karena kita tertarik pada gelombang terhambur (*scattered wave*):

$$\psi_{p,sc}^{(+)}(\mathbf{r}) = A(\mathbf{p}', \mathbf{p}) \frac{1}{(2\pi)^{3/2}} \frac{e^{ipr}}{r} \,. \tag{2}$$

  Lagipula, baik untuk proses A maupun proses B, gelombang datang (*incoming wave*, yaitu suku pertama fungsi gelombang hamburan di Eq. (1)), sama.

  Scattered wave untuk proses B dapat diperoleh dari *scattered wave* untuk proses A dengan mempertukarkan kedua partikel di keadaan akhir (dengan momentum linier $\mathbf{p}'$) memakai operator permutasi P_{12}. Kita hanya perlu mengerjakan operator permutasi untuk amplitudo hamburan $A(\mathbf{p}', \mathbf{p})$:[1]

$$\begin{aligned} A_p(\theta') = A^{(A)}(\mathbf{p}', \mathbf{p}) &= -(2\pi)^2 \mu \langle \mathbf{p}' | V | \mathbf{p} \rangle^{(+)} \\ A^{(B)}(\mathbf{p}', \mathbf{p}) &= -(2\pi)^2 \mu \langle P_{12}\mathbf{p}' | V | \mathbf{p} \rangle^{(+)} \end{aligned} \tag{5}$$

[1] Operator permutasi P_{12} merupakan operator uniter: $P_{12}^{-1} P_{12} = P_{12} P_{12}^{-1} = 1$. Operasi P_{12} 2 kali berturut-turut membawa kembali ke keadaan semula: $P_{12}^2 = P_{12} P_{12} = 1$, sehingga operasi P_{12} sama saja dengan operasi P_{12}^{-1}. Seperti ditunjukkan berikut ini, $P_{12}^{+} = P_{12}^{-1}$:

$$\langle P_{12}\psi | P_{12}\psi \rangle = \langle \psi | \psi \rangle \tag{3}$$

$$\langle P_{12}\psi | P_{12}\psi \rangle = \langle \psi | P_{12}^{+} P_{12} | \psi \rangle \to P_{12}^{+} P_{12} = 1 \to P_{12}^{+} = P_{12}^{-1} \,. \tag{4}$$

$$
\begin{aligned}
&= -(2\pi)^2\mu \langle \mathbf{p}'| P_{12}^{-1} V |\mathbf{p}\rangle^{(+)} \\
&= -(2\pi)^2\mu \langle \mathbf{p}'| P_{12} V |\mathbf{p}\rangle^{(+)} \\
&= -(2\pi)^2\mu \langle -\mathbf{p}'| V |\mathbf{p}\rangle^{(+)} \\
&= A^{(A)}(-\mathbf{p}', \mathbf{p}) \\
&= A_p(\pi - \theta')\,, \qquad (6)
\end{aligned}
$$

sehingga diperoleh *scattered wave* total:

$$
\psi_{p,sc}^{(+)}(\mathbf{r}) = \psi_{p,sc}^{(+)(A)}(\mathbf{r}) + \psi_{p,sc}^{(+)(B)}(\mathbf{r}) = \left(A_p(\theta') + A_p(\pi - \theta')\right) \frac{1}{(2\pi)^{3/2}} \frac{e^{ipr}}{r}\,. \qquad (7)
$$

Dengan demikian, penampang lintang dihitung sebagai:

$$
\frac{d\sigma}{d\Omega} = |A_p(\theta') + A_p(\pi - \theta')|^2\,. \qquad (8)
$$

- Amplitudo hamburan total di Eqs. (7) dihitung sebagai:

$$
\begin{aligned}
A_p^{total}(\theta') &= -(2\pi)^2\mu \langle \mathbf{p}'| V |\mathbf{p}\rangle^{(+)} - (2\pi)^2\mu \langle \mathbf{p}'| P_{12} V |\mathbf{p}\rangle^{(+)} \\
&= -(2\pi)^2\mu \langle \mathbf{p}'| (1 + P_{12}) V |\mathbf{p}\rangle^{(+)}\,. \qquad (9)
\end{aligned}
$$

Perhatikan bahwa Eq. (9) berlaku untuk hamburan tanpa spin (interaksi tidak bergantung pada spin) dan untuk sistem boson, karena keadaan sistem boson bersifat simetrik terhadap pertukaran partikel. Sebaliknya, sistem fermion memiliki keadaan yang bersifat antisimetrik terhadap pertukaran partikel. Dengan demikian, untuk sistem fermion amplitudo hamburan total dihitung sebagai:

$$
A_p^{total}(\theta') = -(2\pi)^2\mu \langle \mathbf{p}'| (1 - P_{12}) V |\mathbf{p}\rangle^{(+)}\,. \qquad (10)
$$

Supaya lebih jelas, untuk sistem boson dan sistem fermion, kita tambahkan keadaan spin $|\lambda_1\rangle$ dan $|\lambda_2\rangle$:

$$
\begin{aligned}
\text{boson}\ :\ |\varphi_{\mathbf{p}_0}\lambda_1\lambda_2\rangle &= \frac{1}{\sqrt{2}}(1 + P_{12})\,|\varphi_{\mathbf{p}_0}\rangle\,|\lambda_1\rangle\,|\lambda_2\rangle \qquad (11) \\
\left|\psi_{\mathbf{p}_0}^{(+)}\lambda_1\lambda_2\right\rangle &= \frac{1}{\sqrt{2}}(1 + P_{12})\,\left|\psi_{\mathbf{p}_0}^{(+)}\right\rangle\,|\lambda_1\rangle\,|\lambda_2\rangle \qquad (12) \\
\text{fermion}\ :\ |\varphi_{\mathbf{p}_0}\lambda_1\lambda_2\rangle &= \frac{1}{\sqrt{2}}(1 - P_{12})\,|\varphi_{\mathbf{p}_0}\rangle\,|\lambda_1\rangle\,|\lambda_2\rangle \qquad (13) \\
\left|\psi_{\mathbf{p}_0}^{(+)}\lambda_1\lambda_2\right\rangle &= \frac{1}{\sqrt{2}}(1 - P_{12})\,\left|\psi_{\mathbf{p}_0}^{(+)}\right\rangle\,|\lambda_1\rangle\,|\lambda_2\rangle \qquad (14)
\end{aligned}
$$

Perhatikan, pada Eqs. (11) dan (12) keadaan sistem boson sudah tersimetrisasi dan pada Eqs. (13) dan (14) keadaan sistem fermion sudah terantisimetrisasi, dengan kata lain keadaan sistem merupakan keadaan eigen operator permutasi P_{12}, seperti dapat ditunjukkan contoh berikut:

$$
\text{boson}\ :\ P_{12}\,|\varphi_{\mathbf{p}_0}\lambda_1\lambda_2\rangle = \frac{1}{\sqrt{2}} P_{12}(1 + P_{12})\,|\varphi_{\mathbf{p}_0}\rangle\,|\lambda_1\rangle\,|\lambda_2\rangle
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}(P_{12}+1)\left|\varphi_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= \left|\varphi_{\mathbf{p}_0}\lambda_1\lambda_2\right\rangle \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\text{fermion} \ : P_{12}\left|\varphi_{\mathbf{p}_0}\lambda_1\lambda_2\right\rangle &= \frac{1}{\sqrt{2}}P_{12}(1-P_{12})\left|\varphi_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= \frac{1}{\sqrt{2}}(P_{12}-1)\left|\varphi_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= -\left|\varphi_{\mathbf{p}_0}\lambda_1\lambda_2\right\rangle\ . \qquad (16)
\end{aligned}
$$

Terapkan Eqs. (11), (12), (13), dan (14) pada amplitudo hamburan:[2]

$$
\begin{aligned}
\text{boson} \ : A^{total}_{p_0,\lambda_1'\lambda_2',\lambda_1\lambda_2}(\theta) &= -(2\pi)^2\mu\hbar\left\langle\varphi_{\mathbf{p}'}\lambda_1'\lambda_2'\right| V\left|\psi^{(+)}_{\mathbf{p}_0}\lambda_1\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\frac{1}{2}\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right|(1+P_{12})V(1+P_{12})\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\frac{1}{2}\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right|(1+P_{12})^2V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right|(1+P_{12})V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right| V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&\quad -(2\pi)^2\mu\hbar\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right| P_{12}V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right| V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&\quad -(2\pi)^2\mu\hbar\left\langle\lambda_2'\right|\left\langle\lambda_1'\right|\left\langle\varphi_{-\mathbf{p}'}\right| V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= A_{p_0,\lambda_1'\lambda_2',\lambda_1\lambda_2}(\theta) + A_{p_0,\lambda_2'\lambda_1',\lambda_1\lambda_2}(\pi-\theta) \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
\text{fermion} \ : A^{total}_{p_0,\lambda_1'\lambda_2',\lambda_1\lambda_2}(\theta) &= -(2\pi)^2\mu\hbar\left\langle\varphi_{\mathbf{p}'}\lambda_1'\lambda_2'\right| V\left|\psi^{(+)}_{\mathbf{p}_0}\lambda_1\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\frac{1}{2}\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right|(1-P_{12})V(1-P_{12})\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\frac{1}{2}\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right|(1-P_{12})^2V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right|(1-P_{12})V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right| V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&\quad +(2\pi)^2\mu\hbar\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right| P_{12}V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= -(2\pi)^2\mu\hbar\left\langle\lambda_1'\right|\left\langle\lambda_2'\right|\left\langle\varphi_{\mathbf{p}'}\right| V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&\quad +(2\pi)^2\mu\hbar\left\langle\lambda_2'\right|\left\langle\lambda_1'\right|\left\langle\varphi_{-\mathbf{p}'}\right| V\left|\psi^{(+)}_{\mathbf{p}_0}\right\rangle\left|\lambda_1\right\rangle\left|\lambda_2\right\rangle \\
&= A_{p_0,\lambda_1'\lambda_2',\lambda_1\lambda_2}(\theta) - A_{p_0,\lambda_2'\lambda_1',\lambda_1\lambda_2}(\pi-\theta)\,. \qquad (19)
\end{aligned}
$$

[2]Interaksi sentral (*central force*) V bersifat invarian terhadap pertukaran partikel:

$$
P_{12}VP_{12}^{-1} = V \rightarrow P_{12}V = VP_{12} \text{ atau } [V,P_{12}]=0 \quad (\text{bukti: } P_{12}VP_{12}^{-1} = VP_{12}P_{12}^{-1} = V)\,. \qquad (17)
$$