

General Formula for Spin Observables of Two Spin Half

1 Hamburan dengan memperhitungkan spin

- Tanpa spin, keadaan hamburan adalah:

$$\begin{aligned}
|\mathbf{p}\rangle^{(+)} &= |\mathbf{p}\rangle + G_0^{(+)}(E_p)V|\mathbf{p}\rangle^{(+)} \\
&= |\mathbf{p}\rangle + G_0^{(+)}(E_p)T|\mathbf{p}\rangle \\
&= |\mathbf{p}\rangle + \int d\mathbf{p}' G_0^{(+)}(E_p)|\mathbf{p}'\rangle\langle\mathbf{p}'|T|\mathbf{p}\rangle \\
&= |\mathbf{p}\rangle + \int d\mathbf{p}' G_0^{(+)}(E_p)|\mathbf{p}'\rangle T(\mathbf{p}', \mathbf{p}) \\
&= |\mathbf{p}\rangle + \lim_{\epsilon \rightarrow 0} \int d\mathbf{p}' \frac{1}{E_p + i\epsilon - H_0} |\mathbf{p}'\rangle T(\mathbf{p}', \mathbf{p}) \\
&= |\mathbf{p}\rangle + \lim_{\epsilon \rightarrow 0} \int d\mathbf{p}' |\mathbf{p}'\rangle \frac{1}{E_p + i\epsilon - E_{p'}} T(\mathbf{p}', \mathbf{p}). \tag{1}
\end{aligned}$$

Ingat kembali, suku ke-2 Eq. (1) memberikan fungsi gelombang terhambur (*scattered outgoing wave*):

$$A(\mathbf{p}', \mathbf{p}) \frac{e^{ipr}}{r}, \quad (\mathbf{p}' = p\hat{\mathbf{p}}'), \tag{2}$$

dengan

$$A(\mathbf{p}', \mathbf{p}) = M(\mathbf{p}', \mathbf{p}) = -\mu(2\pi)^2 T(\mathbf{p}', \mathbf{p}) \tag{3}$$

merupakan amplitudo hamburan.

- Anggap $|n\rangle_i$ menyatakan keadaan murni spin mula-mula (*initial pure spin state*) sistem sebelum hamburan, yang sesuai postulat ekspansi, dapat dinyatakan sebagai:

$$|n\rangle_i = \sum_j \left(a_j^{(n)}\right)_i |\lambda_{1,j}\rangle |\lambda_{2,j}\rangle, \tag{4}$$

dengan $|\lambda_{r,j}\rangle = \left|\frac{1}{2}\lambda_{r,j}\right\rangle$ ($r = 1, 2$) dan $\left(a_j^{(n)}\right)_i$ koefisien ekspansi.¹ Dengan memasukkan spin, keadaan hamburan adalah:

$$|\mathbf{p}\rangle^{(+)}|n\rangle_i = |\mathbf{p}\rangle|n\rangle_i + G_0^{(+)}(E_p)V|\mathbf{p}\rangle^{(+)}|n\rangle_i$$

¹ Eigenstate spin $\frac{1}{2}$:

$$\left|\frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{dan} \quad \left|-\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5}$$

$$\begin{aligned}
&= |\mathbf{p}\rangle|n\rangle_i + G_0^{(+)}(E_p)T|\mathbf{p}\rangle|n\rangle_i \\
&= |\mathbf{p}\rangle|n\rangle_i + G_0^{(+)}(E_p)T|\mathbf{p}\rangle \sum_j \left(a_j^{(n)}\right)_i |\lambda_{1,j}\rangle|\lambda_{2,j}\rangle \\
&= |\mathbf{p}\rangle|n\rangle_i + \sum_j G_0^{(+)}(E_p)T|\mathbf{p}\rangle|\lambda_{1,j}\rangle|\lambda_{2,j}\rangle \left(a_j^{(n)}\right)_i \\
&= |\mathbf{p}\rangle|n\rangle_i + \sum_k \int d\mathbf{p}' \sum_j G_0^{(+)}(E_p)|\lambda_{1,k}\rangle|\lambda_{2,k}\rangle\langle\lambda_{1,k}|\langle\lambda_{2,k}|\mathbf{p}'\rangle\langle\mathbf{p}'|T|\mathbf{p}\rangle|\lambda_{1,j}\rangle|\lambda_{2,j}\rangle \left(a_j^{(n)}\right)_i \\
&= |\mathbf{p}\rangle|n\rangle_i + \sum_k \int d\mathbf{p}' \sum_j G_0^{(+)}(E_p)|\mathbf{p}'\rangle|\lambda_{1,k}\rangle|\lambda_{2,k}\rangle\langle\lambda_{1,k}|\langle\lambda_{2,k}|\langle\mathbf{p}'|T|\mathbf{p}\rangle|\lambda_{1,j}\rangle|\lambda_{2,j}\rangle \left(a_j^{(n)}\right)_i \\
&= |\mathbf{p}\rangle|n\rangle_i + \sum_k \int d\mathbf{p}' G_0^{(+)}(E_p)|\mathbf{p}'\rangle|\lambda_{1,k}\rangle|\lambda_{2,k}\rangle \sum_j T_{kj}(\mathbf{p}', \mathbf{p}) \left(a_j^{(n)}\right)_i \\
&= |\mathbf{p}\rangle|n\rangle_i + \lim_{\epsilon \rightarrow 0} \sum_k \int d\mathbf{p}' \frac{1}{E_p + i\epsilon - H_0} |\mathbf{p}'\rangle|\lambda_{1,k}\rangle|\lambda_{2,k}\rangle \sum_j T_{kj}(\mathbf{p}', \mathbf{p}) \left(a_j^{(n)}\right)_i \\
&= |\mathbf{p}\rangle|n\rangle_i + \lim_{\epsilon \rightarrow 0} \sum_k \int d\mathbf{p}' |\mathbf{p}'\rangle|\lambda_{1,k}\rangle|\lambda_{2,k}\rangle \frac{1}{E_p + i\epsilon - E_{p'}} \sum_j T_{kj}(\mathbf{p}', \mathbf{p}) \left(a_j^{(n)}\right)_i.
\end{aligned} \tag{7}$$

Untuk singkatnya tanpa ditunjukkan secara detil, bandingkan Eq. (7) dengan Eqs. (1), (2), dan (3), kita dapatkan fungsi gelombang terhambur:

$$\sum_k A_k(\mathbf{p}', \mathbf{p}) \frac{e^{ipr}}{r} |\lambda_{1,k}\rangle|\lambda_{2,k}\rangle, \quad (\mathbf{p}' = p\hat{\mathbf{p}}'), \tag{8}$$

dengan amplitudo hamburan untuk tiap keadaan akhir spin $|\lambda_{1,k}\rangle|\lambda_{2,k}\rangle$:

$$A_k(\mathbf{p}', \mathbf{p}) = \sum_j M_{kj}(\mathbf{p}', \mathbf{p}) \left(a_j^{(n)}\right)_i = -\mu(2\pi)^2 \sum_j T_{kj}(\mathbf{p}', \mathbf{p}) \left(a_j^{(n)}\right)_i. \tag{9}$$

- Anggap peluang mendapatkan $|n\rangle_i$ dalam *initial mixed state* adalah P_n , maka *initial density Direct product / outer product / tensor product* $|\lambda_{1,j}\rangle|\lambda_{2,j}\rangle = |\lambda_{1,j}\rangle \otimes |\lambda_{2,j}\rangle$ adalah:

$$\begin{aligned}
\left|\frac{1}{2}\right\rangle \left|\frac{1}{2}\right\rangle &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \left|\frac{1}{2}\right\rangle \left|-\frac{1}{2}\right\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\left|-\frac{1}{2}\right\rangle \left|\frac{1}{2}\right\rangle &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \left|-\frac{1}{2}\right\rangle \left|-\frac{1}{2}\right\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned} \tag{6}$$

matrik ρ_i adalah:

$$\begin{aligned}\rho_i &= \sum_n |n\rangle_i P_n |n\rangle_i \\ &= \sum_{kl} |\lambda_{1,k}\rangle |\lambda_{2,k}\rangle \sum_n \left(a_k^{(n)}\right)_i P_n \left(a_l^{(n)*}\right)_i \langle\lambda_{1,l}| \langle\lambda_{2,l}| \\ &= \sum_{kl} |\lambda_{1,k}\rangle |\lambda_{2,k}\rangle (\rho_i)_{kl} \langle\lambda_{1,l}| \langle\lambda_{2,l}| \end{aligned}\quad (10)$$

$$\text{dengan } (\rho_i)_{kl} = \sum_n \left(a_k^{(n)}\right)_i P_n \left(a_l^{(n)*}\right)_i. \quad (11)$$

- Anggap $|n\rangle_f$ menyatakan keadaan murni spin akhir (*final pure spin state*) sistem setelah hamburan dengan peluang P_n :

$$|n\rangle_f = \sum_k \left(a_k^{(n)}\right)_f |\lambda_{1,k}\rangle |\lambda_{2,k}\rangle. \quad (12)$$

Perbedaan keadaan akhir $|n\rangle_f$ di Eq. (12) dari keadaan awal $|n\rangle_i$ di Eq. (4) ada di koefisien ekspansi. Perubahan koefisien ekspansi dari $\left(a_j^{(n)}\right)_i$ menjadi $\left(a_k^{(n)}\right)_f$, seperti dapat dilihat dari fungsi gelombang terhambur di Eqs. (8) dan (9), ditentukan oleh matriks hamburan (*scattering matrix*) $M = -\mu(2\pi)^2 T$ sebagai berikut:

$$\left(a_k^{(n)}\right)_f = \sum_j M_{kj} \left(a_j^{(n)}\right)_i. \quad (13)$$

Perhatikan bahwa sesuai Eq. (13) $\left(a_k^{(n)}\right)_f$ bergantung pada sudut hambur $\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}$, sebagaimana M bergantung pada $\hat{\mathbf{p}}' \cdot \hat{\mathbf{p}}$. *Final density matrix* ρ_f menjadi:

$$\begin{aligned}\rho_f &= \sum_n |n\rangle_f P_n |n\rangle_f \\ &= \sum_{kl} |\lambda_{1,k}\rangle |\lambda_{2,k}\rangle \sum_n \left(a_k^{(n)}\right)_f P_n \left(a_l^{(n)*}\right)_f \langle\lambda_{1,l}| \langle\lambda_{2,l}| \\ &= \sum_{kl} |\lambda_{1,k}\rangle |\lambda_{2,k}\rangle \sum_n \sum_\alpha M_{k\alpha} \left(a_\alpha^{(n)}\right)_i P_n \sum_\beta M_{l\beta}^* \left(a_\beta^{(n)*}\right)_i \langle\lambda_{1,l}| \langle\lambda_{2,l}| \\ &= \sum_{kl} |\lambda_{1,k}\rangle |\lambda_{2,k}\rangle \sum_{\alpha\beta} M_{k\alpha} \sum_n \left(a_\alpha^{(n)}\right)_i P_n \left(a_\beta^{(n)*}\right)_i M_{l\beta}^* \langle\lambda_{1,l}| \langle\lambda_{2,l}| \\ &= \sum_{kl} |\lambda_{1,k}\rangle |\lambda_{2,k}\rangle \sum_{\alpha\beta} M_{k\alpha} (\rho_i)_{\alpha\beta} M_{\beta l}^+ \langle\lambda_{1,l}| \langle\lambda_{2,l}| \\ &= \sum_{kl} |\lambda_{1,k}\rangle |\lambda_{2,k}\rangle (\rho_f)_{kl} \langle\lambda_{1,l}| \langle\lambda_{2,l}| \end{aligned}\quad (14)$$

$$\rightarrow (\rho_f)_{kl} = \sum_{\alpha\beta} M_{k\alpha} (\rho_i)_{\alpha\beta} M_{\beta l}^+ \quad (15)$$

$$\begin{aligned}
&= \sum_{\alpha\beta} \langle \lambda_{1,k} | \langle \lambda_{2,k} | M | \lambda_{1,\alpha} \rangle | \lambda_{2,\alpha} \rangle \langle \lambda_{1,\alpha} | \langle \lambda_{2,\alpha} | \rho_i | \lambda_{1,\beta} \rangle | \lambda_{2,\beta} \rangle \langle \lambda_{1,\beta} | \langle \lambda_{2,\beta} | M^+ | \lambda_{1,l} \rangle | \lambda_{2,l} \rangle \\
&= \langle \lambda_{1,k} | \langle \lambda_{2,k} | M \sum_{\alpha} | \lambda_{1,\alpha} \rangle | \lambda_{2,\alpha} \rangle \langle \lambda_{1,\alpha} | \langle \lambda_{2,\alpha} | \rho_i \sum_{\beta} | \lambda_{1,\beta} \rangle | \lambda_{2,\beta} \rangle \langle \lambda_{1,\beta} | \langle \lambda_{2,\beta} | M^+ | \lambda_{1,l} \rangle | \lambda_{2,l} \rangle \\
&= \langle \lambda_{1,k} | \langle \lambda_{2,k} | M \rho_i M^+ | \lambda_{1,l} \rangle | \lambda_{2,l} \rangle \\
&= \langle \lambda_{1,k} | \langle \lambda_{2,k} | \rho_f | \lambda_{1,l} \rangle | \lambda_{2,l} \rangle \\
&\rightarrow \rho_f = M \rho_i M^+. \tag{16}
\end{aligned}$$

- Penampang lintang differensial untuk keadaan akhir spin $|\lambda_{1,k}\rangle|\lambda_{2,k}\rangle$ untuk kasus *pure state*:

$$\left\langle \frac{d\sigma_k}{d\Omega} \right\rangle_n = |A_k(\mathbf{p}', \mathbf{p})|^2 = \left| \sum_j M_{kj}(\mathbf{p}', \mathbf{p}) \left(a_j^{(n)} \right)_i \right|^2 = \left| \left(a_k^{(n)} \right)_f \right|^2. \tag{17}$$

- Penampang lintang differensial untuk keadaan akhir spin $|\lambda_{1,k}\rangle|\lambda_{2,k}\rangle$ untuk kasus *mixed state* (keadaan yang sebenarnya / riil):

$$\begin{aligned}
I &= \frac{\sum_n P_n \left\langle \frac{d\sigma_k}{d\Omega} \right\rangle_n}{\sum_n P_n \langle n | n \rangle} \\
&= \frac{\sum_n P_n \left| \left(a_k^{(n)} \right)_f \right|^2}{\sum_n P_n \sum_j \left| \left(a_j^{(n)} \right)_i \right|^2} \\
&= \frac{\sum_n \left(a_k^{(n)} \right)_f P_n \left(a_k^{(n)*} \right)_f}{\sum_j \sum_n \left(a_j^{(n)} \right)_i P_n \left(a_j^{(n)*} \right)_i} \\
&= \frac{(\rho_f)_{kk}}{\sum_j (\rho_i)_{jj}} \\
&= \frac{(\rho_f)_{kk}}{\text{Tr} (\rho_i)}. \tag{18}
\end{aligned}$$

- Penampang lintang differensial untuk sembarang keadaan akhir spin $|\lambda_{1,k}\rangle|\lambda_{2,k}\rangle$ untuk kasus *mixed state* (keadaan yang sebenarnya / riil):

$$I = \sum_k \left(\frac{\sum_n P_n \left\langle \frac{d\sigma_k}{d\Omega} \right\rangle_n}{\sum_n P_n \langle n | n \rangle} \right) = \frac{\sum_k (\rho_f)_{kk}}{\text{Tr} (\rho_i)} = \frac{\text{Tr} (\rho_f)}{\text{Tr} (\rho_i)}. \tag{19}$$

2 Operator spin, *density matrix*, dan rumus umum besaran spin untuk 2 spin $\frac{1}{2}$

- Ambillah $|n\rangle$ sebagai sembarang keadaan murni (*pure state*) spin s :

$$|n\rangle = \sum_i a_i^{(n)} |\lambda_{1,i}\rangle |\lambda_{2,i}\rangle, \quad (20)$$

dengan $|\lambda_{r,i}\rangle = |\frac{1}{2}\lambda_{r,i}\rangle$ ($r = 1, 2$) dan $a_i^{(n)}$ koefisien ekspansi.

- Anggaplah suatu sistem berada dalam suatu keadaan, yang merupakan campuran keadaan murni $|n\rangle$, masing-masing dengan peluang P_n . Hasil pengukuran suatu besaran spin $\overline{\langle O \rangle}$ pada sistem tersebut adalah:

$$\overline{\langle O \rangle} = \frac{\sum_n P_n \langle n | \hat{O} | n \rangle}{\sum_n P_n \langle n | n \rangle} = \frac{\text{Tr}(\rho O)}{\text{Tr}(\rho)} = \frac{\text{Tr}(O\rho)}{\text{Tr}(\rho)}, \quad (21)$$

dengan ρ *density matrix*:

$$\rho = \sum_n |n\rangle P_n \langle n| = \sum_{ji} |\lambda_{1,j}\rangle |\lambda_{2,j}\rangle \rho_{ji} \langle \lambda_{1,i}| \langle \lambda_{2,i}|, \quad (22)$$

dan elemen matriksnya:

$$\rho_{ji} = \langle \lambda_{1,j} | \langle \lambda_{2,j} | \rho | \lambda_{1,i} \rangle | \lambda_{2,i} \rangle = \sum_n a_j^{(n)} P_n a_i^{(n)*}. \quad (23)$$

- Operator untuk mengukur bermacam-macam besaran spin \hat{O} seperti apa? Untuk sistem 2 spin $\frac{1}{2}$, pilihan yang logis adalah \hat{O} sama dengan *direct product / outer product / tensor product* dua operator spin $\frac{1}{2}$, yaitu $\hat{O} = \sigma_\mu^{(1)} \otimes \sigma_\nu^{(2)}$ ($\mu, \nu = 0, 1, 2, 3$), kita singkat $\hat{O} = \sigma_0^{(1)} \sigma_3^{(2)}$, dengan σ_0 matriks identitas berukuran 2×2 dan $\sigma_1, \sigma_2, \sigma_3$ adalah matriks Pauli:²

$$\begin{aligned} \sigma_\mu^{(1)}, \sigma_\nu^{(2)} &= \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \end{aligned} \quad (26)$$

²Contoh:

$$\sigma_0^{(1)} \sigma_3^{(2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (24)$$

$$\sigma_2^{(1)} \sigma_3^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad (25)$$

Nilai suatu besaran spin $\overline{\langle O \rangle}$ menjadi (lihat Eq. (21)):

$$\overline{\langle O \rangle} = \overline{\left\langle \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right\rangle} = \frac{\text{Tr} \left(\rho \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right)}{\text{Tr} (\rho)} = \frac{\text{Tr} \left(\sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \rho \right)}{\text{Tr} (\rho)}, \quad (\mu, \nu = 0, 1, 2, 3). \quad (27)$$

- *Density matrix* ρ seperti apa? Untuk sistem 2 spin $\frac{1}{2}$, pilihan yang logis adalah ρ merupakan kombinasi linier $\sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)}$ ($\mu, \nu = 0, 1, 2, 3$):

$$\rho = \sum_{\mu, \nu=0}^3 a_{\mu\nu} \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)}, \quad (28)$$

dengan $a_{\mu\nu}$ koefisien kombinasi linier, diperoleh dengan menerapkan *orthogonality* $\sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)}$:

$$\text{Tr} \left(\sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \sigma_{\alpha}^{(1)} \sigma_{\beta}^{(2)} \right) = \text{Tr} \left(\sigma_{\alpha}^{(1)} \sigma_{\beta}^{(2)} \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right) = 4\delta_{\mu\alpha}\delta_{\nu\beta}, \quad (29)$$

sehingga:

$$\begin{aligned} \text{Tr} \left(\sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \rho \right) &= \text{Tr} \left(\rho \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right) = \sum_{\alpha, \beta=0}^3 a_{\alpha\beta} \text{Tr} \left(\sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \sigma_{\alpha}^{(1)} \sigma_{\beta}^{(2)} \right) = \sum_{\alpha, \beta=0}^3 a_{\alpha\beta} 4\delta_{\mu\alpha}\delta_{\nu\beta} = 4a_{\mu\nu} \\ \rightarrow a_{\mu\nu} &= \frac{1}{4} \text{Tr} \left(\sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \rho \right) = \frac{1}{4} \text{Tr} \left(\rho \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right). \end{aligned} \quad (30)$$

Dengan demikian, *density matrix* ρ dapat ditulis sebagai:

$$\rho = \frac{1}{4} \sum_{\mu, \nu=0}^3 \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \text{Tr} \left(\rho \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right) = \frac{1}{4} \sum_{\mu, \nu=0}^3 \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \text{Tr} \left(\sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \rho \right). \quad (31)$$

Kemudian, dari Eq. (27) diperoleh:

$$\text{Tr} \left(\rho \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right) = \text{Tr} \left(\sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \rho \right) = \overline{\left\langle \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right\rangle} \text{Tr} (\rho), \quad (32)$$

sehingga ρ dapat ditulis sebagai:

$$\rho = \frac{1}{4} \text{Tr} (\rho) \sum_{\mu, \nu=0}^3 \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \overline{\left\langle \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right\rangle}. \quad (33)$$

- Pada eksperimen hamburan orang *set up* keadaan awal, kemudian jalankan proses hamburan, dan pada keadaan akhir orang ukur besaran-besaran spin. Terapkan Eq. (27) untuk keadaan akhir serta gunakan Eqs. (16), (19), dan (33):

$$\overline{\left\langle \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right\rangle}_f = \frac{\text{Tr} \left(\rho_f \sigma_{\mu}^{(1)} \sigma_{\nu}^{(2)} \right)}{\text{Tr} (\rho_f)}$$

$$\begin{aligned}
&= \frac{\text{Tr} \left(M \rho_i M^+ \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right)}{\text{Tr} (\rho_f)} \\
&= \frac{\text{Tr} \left(M \frac{1}{4} \text{Tr} (\rho_i) \sum_{\alpha,\beta=0}^3 \sigma_\alpha^{(1)} \sigma_\beta^{(2)} \overline{\left\langle \sigma_\alpha^{(1)} \sigma_\beta^{(2)} \right\rangle}_i M^+ \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right)}{\text{Tr} (\rho_f)} \\
&= \frac{1}{4} \frac{\text{Tr} (\rho_i)}{\text{Tr} (\rho_f)} \sum_{\alpha,\beta=0}^3 \overline{\left\langle \sigma_\alpha^{(1)} \sigma_\beta^{(2)} \right\rangle}_i \text{Tr} \left(M \sigma_\alpha^{(1)} \sigma_\beta^{(2)} M^+ \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right) \\
&= \frac{1}{4I} \sum_{\alpha,\beta=0}^3 \overline{\left\langle \sigma_\alpha^{(1)} \sigma_\beta^{(2)} \right\rangle}_i \text{Tr} \left(M \sigma_\alpha^{(1)} \sigma_\beta^{(2)} M^+ \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right) \\
\rightarrow I \overline{\left\langle \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right\rangle}_f &= \frac{1}{4} \sum_{\alpha,\beta=0}^3 \overline{\left\langle \sigma_\alpha^{(1)} \sigma_\beta^{(2)} \right\rangle}_i \text{Tr} \left(M \sigma_\alpha^{(1)} \sigma_\beta^{(2)} M^+ \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right). \tag{34}
\end{aligned}$$

Persamaan (34) adalah rumus umum besaran spin untuk 2 spin $\frac{1}{2}$. Rumus tersebut menghubungkan keadaan awal spin $\overline{\left\langle \sigma_\alpha^{(1)} \sigma_\beta^{(2)} \right\rangle}_i$ dan keadaan akhir spin $\overline{\left\langle \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right\rangle}_f$. $\sigma_\alpha^{(1)} \sigma_\beta^{(2)}$ dan $\sigma_\mu^{(1)} \sigma_\nu^{(2)}$ masing-masing adalah operator untuk mengukur spin di keadaan awal dan akhir. Faktor $\frac{1}{4}$ didapat dari jumlah keadaan yang mungkin untuk 2 spin $\frac{1}{2}$, yaitu 4. Dalam T -matrix rumus itu dinyatakan sebagai:

$$I \overline{\left\langle \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right\rangle}_f = \frac{1}{4} (4\pi^2 \mu)^2 \sum_{\alpha,\beta=0}^3 \overline{\left\langle \sigma_\alpha^{(1)} \sigma_\beta^{(2)} \right\rangle}_i \text{Tr} \left(T \sigma_\alpha^{(1)} \sigma_\beta^{(2)} T^+ \sigma_\mu^{(1)} \sigma_\nu^{(2)} \right). \tag{35}$$

- Contoh, *spin-averaged differential cross section*. Pada eksperimen ini keadaan spin awal sistem acak (*unpolarized*) ($\alpha = \beta = 0$, $\overline{\left\langle \sigma_0^{(1)} \sigma_0^{(2)} \right\rangle}_i = 1$, $\overline{\left\langle \sigma_{\alpha \neq 0}^{(1)} \sigma_{\beta \neq 0}^{(2)} \right\rangle}_i = 0$) dan pada keadaan akhir tidak dilakukan pengukuran spin ($\mu = \nu = 0$, $\overline{\left\langle \sigma_0^{(1)} \sigma_0^{(2)} \right\rangle}_f = 1$):

$$I = I_0 = \frac{1}{4} \text{Tr} (MM^+) = \frac{1}{4} (4\pi^2 \mu)^2 \text{Tr} (TT^+). \tag{36}$$

Penampang lintang di Eq. (36) adalah penampang lintang di Eq. (19), namun dibagi jumlah keadaan spin yang mungkin untuk 2 spin $\frac{1}{2}$, yaitu 4:

$$I_0 = \frac{1}{4} \frac{\text{Tr} (\rho_f)}{\text{Tr} (\rho_i)} = \frac{1}{4} \frac{\text{Tr} (M \rho_i M^+)}{\text{Tr} (\rho_i)} = \frac{1}{4} \text{Tr} (MM^+). \tag{37}$$

Pada Eq. (37) $\rho_i = \sigma_0^{(1)} \sigma_0^{(2)}$, karena keadaan spin *unpolarized*, tidak ada *preferred* spin pada arah tertentu (pada Eq. (28) $a_{00} = 1$ dan $a_{\mu \neq 0, \nu \neq 0} = 0$).