phase and the scattering amplitude has its maximum value:

$$A_{ba}^{(B)}(0) = -\frac{\mu}{2\pi\hbar^2}\int V(\boldsymbol{r})\, d^3\boldsymbol{r}.$$

In other directions of scattering the contributions from different volume elements differ in phase. The effect of the interference of waves, scattered by different volume elements, can be taken into account by the ratio

$$F(\boldsymbol{q}) = \frac{A_{ba}^{(B)}(\boldsymbol{q})}{A_{ba}^{(B)}(0)},$$

which is usually called the *form factor.*

96*. The Free-particle Green Function

The Green function for the free motion of a particle is determined by equation (95.5). We write this equation in the form

$$G(\boldsymbol{r}|\boldsymbol{r}') = (\nabla^2 + k^2)^{-1}\,\delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{96.1}$$

Substituting into (96.1) the integral representation of the δ-function in terms of the eigenfunctions of the operator of a free particle,

$$\delta(\boldsymbol{r} - \boldsymbol{r}') = (2\pi)^{-3}\int e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\boldsymbol{r}')}\, d^3\boldsymbol{q},$$

we find

$$G(\boldsymbol{r}|\boldsymbol{r}') = G(\boldsymbol{r} - \boldsymbol{r}') = \frac{1}{(2\pi)^3}\int \frac{e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\boldsymbol{r}')}}{k^2 - q^2}\, d^3\boldsymbol{q}. \tag{96.1a}$$

We can integrate over the angles and get

$$G(x) = \frac{1}{4\pi^2 ix}\int_{-\infty}^{+\infty}\frac{qe^{iqx}\,dq}{k^2 - q^2}, \tag{96.2}$$

with $x = |\boldsymbol{r} - \boldsymbol{r}'|$.

We can evaluate the integral in (96.2) using the residue theorem. Its value remains undefined until we have determined how to go round the poles $q = \pm k$. The rules for going round the poles follow from the boundary conditions imposed upon $G(x)$ as $x \to \infty$. To find the solution corresponding to outgoing waves, we must choose the path of integration labelled A in Fig. 16. The integral (96.2) is then equal to $2\pi i$ times the residue of the integrand at the only pole $q = k$ which lies inside the contour of integration. We find thus

$$G_{(+)}(x) = -\frac{e^{ikx}}{4\pi x}. \tag{96.3}$$

To find the Green function $G_{(-)}(x)$ corresponding to incoming waves, we must integrate (96.2) over the contour B of Fig. 16. In that case, the pole $q = -k$ will be inside the contour and we have

$$G_{(-)}(x) = -\frac{e^{-ikx}}{4\pi x}. \tag{96.4}$$

The rule for going round the poles can also be found formally by replacing k by $k + i\varepsilon$ in the denominator in (96.2) for the case of $G_{(+)}(x)$, where ε is a small positive quantitiy which after the integral has been evaluated must be made to vanish. If we make this substitution, the poles of the integrand $q = \pm(k + i\varepsilon)$ are shifted into the

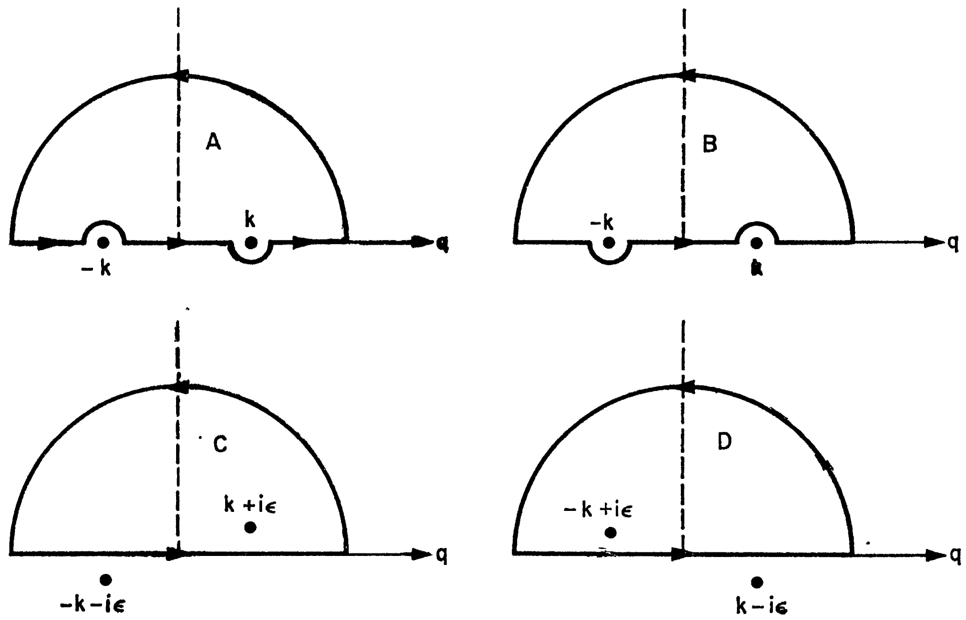

FIG. 16. Rule for going around poles the to get the Green functions $G_{(+)}$ and $G_{(-)}$.

complex plane (Fig. 16C) and inside the contours we have only the pole $k + i\varepsilon$. After integration, we must take the limit $\varepsilon \to 0$. To obtain $G_{(-)}(x)$ we must replace k by $k - i\varepsilon$ in the integrand in (96.2) (Fig. 16D).

In several cases, it is not necessary to have the explicit form of the Green function in intermediate calculations and it is convenient to use a symbolical notation. We shall use equation (95.1) to illustrate how this is done.

To be able to generalise later one, we write equation (95.1) in the form

$$[E_a - \hat{H}_0]\,\psi = \hat{V}\psi, \tag{96.5}$$

where

$$\hat{H}_0 = -\frac{\hbar^2}{2\mu}\nabla^2 \tag{96.6}$$

is the operator of a free particle with reduced mass μ and E_a is the energy of the relative motion. The formal solution of equation (96.5) corresponding to an "incoming" wave φ_a satisfying the equation

$$[E_a - \hat{H}_0]\,\varphi_a = 0 \tag{96.7}$$

will be

$$\psi_a = \varphi_a + [E_a - \hat{H}_0]^{-1}\,\hat{V}\psi_a.$$

To find the solution which contains only an outgoing scattered wave, we must indicate how to go around the poles corresponding to the energy E_a. This can be done most easily by replacing E_a by the complex quantity $E_a + i\varepsilon$. The required solution then has the form

$$\psi_a^{(+)} = \varphi_a + [E_a + i\varepsilon - \hat{H}_0]^{-1} \hat{V}\psi_a^{(+)}. \tag{96.8}$$

The solution of equation (96.5) corresponding to incoming waves will be determined by the equation

$$\psi_a^{(-)} = \varphi_a + [E_a - i\varepsilon - \hat{H}_0]^{-1} \hat{V}\psi_a^{(-)}. \tag{96.9}$$

Equations (96.8) and (96.9) are integral equations. To write equation (96.8) out in explicit form, we must expand the function $\hat{V}\psi_a^{(+)}$ in terms of the eigenfunctions φ_q of the operator $\hat{H}_0$, that is, in terms of the functions satisfying the equation

$$[E_q - \hat{H}_0]\,\varphi_q = 0. \tag{96.10}$$

In our case, the operator $\hat{H}_0$ is the operator of the kinetic energy of a free particle and its eigenfunctions are plane waves, normalised in q-space:

$$\varphi_q = (2\pi)^{-3/2}\, e^{i(\boldsymbol{q}\cdot\boldsymbol{r})}, \quad E_q = \frac{\hbar^2 q^2}{2\mu}. \tag{96.10a}$$

Expanding $\hat{V}\psi_a^{(+)}$ in terms of the complete orthonormal systems of functions φ_q we have

$$\hat{V}\psi_a^{(+)} = \int \varphi_q \langle \varphi_q | \hat{V} | \psi_a^{(+)} \rangle\, d^3\boldsymbol{q}, \tag{96.11}$$

where

$$\langle \varphi_q | \hat{V} | \psi_a^{(+)} \rangle = (2\pi)^{-3/2} \int e^{-i(\boldsymbol{q}\cdot\boldsymbol{r}')} V(\boldsymbol{r}')\, \psi_a^{(+)}(\boldsymbol{r}')\, d^3\boldsymbol{r}'. \tag{96.12}$$

Substituting (96.11) into equation (96.8) and using the fact that the φ_q are the eigenfunctions of the operator $\hat{H}_0$ (see (96.10)), we can write

$$\psi_a^{(+)}(\boldsymbol{r}) = \varphi_a(\boldsymbol{r}) + \int \frac{\varphi_q \langle \varphi_q | \hat{V} | \psi_a^{(+)} \rangle}{E_a + i\varepsilon - E_q}\, d^3\boldsymbol{q}.$$

Substituting (96.10a), (96.12) and $E_a = \hbar^2 k^2/2\mu$ into this equation, we find the explicit form of the integral equation

$$\psi_a^{(+)}(\boldsymbol{r}) = \varphi_a(\boldsymbol{r}) + \frac{2\mu}{\hbar^2(2\pi)^3} \int \frac{V(\boldsymbol{r}')\, \psi_a^{(+)}(\boldsymbol{r}')\, e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\boldsymbol{r}')}\, d^3\boldsymbol{q}\, d^3\boldsymbol{r}'}{(k + i\varepsilon')^2 - q^2}, \tag{96.13}$$

where $\varepsilon' = \mu\varepsilon/\hbar^2 k$. Using the fact that

$$(2\pi)^{-3} \int \frac{e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\boldsymbol{r}')}}{(k + i\varepsilon')^2 - q^2}\, d^3\boldsymbol{q} = G_{(+)}(\boldsymbol{r} - \boldsymbol{r}')$$

and (96.3), we see that equation (96.13) is exactly the same as the integral equation (95.8).