CHAPTER XI

QUANTUM THEORY OF SCATTERING

95. Elastic Scattering of Spin-zero Particles

It is well known from classical mechanics that in the non-relativistic approximation, the problem of the scattering of a particle of mass m_1 by a particle of mass m_2, when the interaction $V(\boldsymbol{r})$ between the particles depends on the relative coordinate $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$, can be reduced to the problem of the scattering of a fictitious particle with the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ in a potential field $V(\boldsymbol{r})$. This reduction of the problem of the elastic scattering of two particles to the motion of a fictitious particle with the reduced mass μ in the potential field $V(\boldsymbol{r})$ is realised by the simple change to a system of coordinates fixed in the centre of mass of the colliding particles. In the following, we shall use only the centre of mass system.

Elastic scattering is the name of scattering in which the internal states and structures of the colliding particles do not change. The initial stage of the scattering process is the motion towards each other of two particles at infinite distance from one another (Fig. 15). When they approach each other, the interaction between the particles changes their motion and after that the particles fly away. The final stage of the scattering process is when the particles move away from one another.

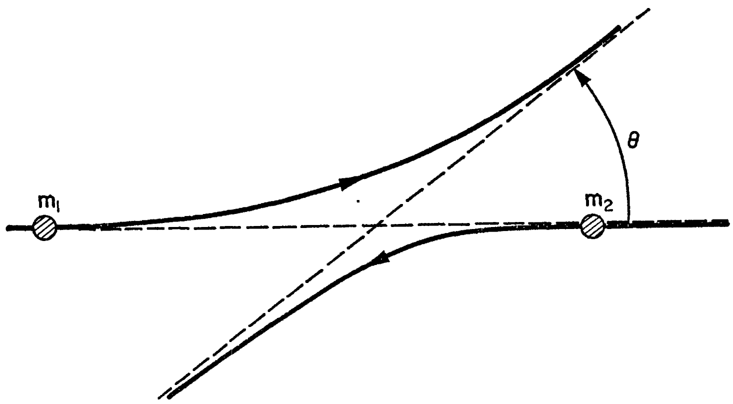

Fig. 15. Scattering in the centre of mass system; θ: scattering angle.

It is often convenient to consider instead of a temporal description an equivalent stationary problem. When we use such a stationary description, we assume that there is a continuous current of particles coming in from infinity which changes because of the interaction with the scattering centre into a current of outgoing (scattered) particles. The scattering problem consists in the evaluation for a given force field of the current of the scattered particles—at infinite distance from the scattering centre—as a function of the current of the incoming particles.

Since the scattered particles move as free particles at large distances from the centre, the energy of their relative motion is always positive and not quantised. We are thus dealing with the continuous spectrum in the scattering problem. Thus, in the stationary formulation the scattering problem of a particle of mass μ with positive relative energy E in a potential field $V(\boldsymbol{r})$ reduces to solving the Schrödinger equation

$$(\nabla^2 + k^2)\,\psi(\boldsymbol{r}) = \frac{2\mu\hat{V}(\boldsymbol{r})}{\hbar^2}\,\psi(\boldsymbol{r}) \tag{95.1}$$

with

$$k^2 = 2\mu E/\hbar^2. \tag{95.2}$$

We shall assume that $V(\boldsymbol{r})$ is non-vanishing in a limited region of space, $|\boldsymbol{r}| \leqq d$. We shall call this part of space the *range of the force* or the *scattering region.* Outside the range of the force, the particles move freely and their state can be a plane wave

$$\varphi_a(\boldsymbol{r}) = \exp i(\boldsymbol{k}_a \cdot \boldsymbol{r}), \quad \boldsymbol{k}_a^2 = \boldsymbol{k}^2, \tag{95.3}$$

satisfying the wave equation (95.1) without its right-hand side. The wave-vector $\boldsymbol{k}_a$ is connected with the momentum $\boldsymbol{p}$ of the relative motion by the simple relation $\boldsymbol{p} = \hbar\boldsymbol{k}_a$. We normalised the function $\varphi_a(\boldsymbol{r})$ by requiring the particle flux density to be numerically equal to the velocity of the relative motion, that is,

$$\boldsymbol{j}_a = \frac{\hbar}{2\mu i}(\varphi_a^*\nabla\varphi_a - \varphi_a\nabla\varphi_a^*) = \frac{\hbar\boldsymbol{k}_a}{\mu}. \tag{95.4}$$

Let $\boldsymbol{j}_a$ describe the flux of the "incoming" particles whose state corresponds to the plane wave (95.3). The particles are scattered because of the interaction. Our problem consists in finding those solutions of equation (95.1) which can be written as a superposition of the plane wave (95.3) and scattered waves coming from the region of the range of the force. We can easily obtain such a solution by using the *Green function* of the operator of the left-hand side of equation (95.1), which is the operator for the motion of a free particle. The Green function of a free particle is the function $G(\boldsymbol{r}|\boldsymbol{r}')$, which satisfies the equation with a point source

$$(\nabla^2 + k^2)\,G(\boldsymbol{r}|\boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{95.5}$$

If we know the solution of equation (95.5), we can always write the general solution of the equation

$$(\nabla^2 + k^2)\,\Phi(\boldsymbol{r}) = A(\boldsymbol{r}) \tag{95.6}$$

in the form

$$\Phi(\boldsymbol{r}) = \varphi(\boldsymbol{r}) + \int G(\boldsymbol{r}|\boldsymbol{r}')\,A(\boldsymbol{r}')\,d^3r', \tag{95.6a}$$

where $\varphi(\boldsymbol{r})$ is a solution of equation (95.6) without its right-hand side.

We shall show in the next section that the solution of equation (95.5) corresponding to an outgoing—scattered—wave is of the form

$$G_{(+)}(\boldsymbol{r}|\boldsymbol{r}') = -\frac{\exp(ik|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}, \tag{95.7}$$

so that we can use (95.6) and (95.6a) to transform equation (95.1) as follows

$$\psi_a(\boldsymbol{r}) = \varphi_a(\boldsymbol{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{\exp(ik|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|} V(\boldsymbol{r}')\,\psi_a(\boldsymbol{r}')\,d^3\boldsymbol{r}'. \tag{95.8}$$

This is an integral equation which determines the complete wavefunction ψ_a of the scattering problem.

At large distances—$r \gg d$—we can write $k|\boldsymbol{r}-\boldsymbol{r}'| \approx kr - (\boldsymbol{k}_b \cdot \boldsymbol{r}')$, where $\boldsymbol{k}_b = k\boldsymbol{r}/r$; the asymptotic form of $\psi_a(\boldsymbol{r})$ is thus

$$\psi_a(\boldsymbol{r}) = \varphi_a(\boldsymbol{r}) + A_{ba}\frac{e^{ikr}}{r}, \quad r \gg d, \tag{95.9}$$

with

$$A_{ba} = -\frac{\mu}{2\pi\hbar^2}\int e^{-i(\boldsymbol{k}_b\cdot\boldsymbol{r}')}V(\boldsymbol{r}')\,\psi_a(\boldsymbol{r}')\,d^3\boldsymbol{r}'. \tag{95.10}$$

If we bear in mind that $\varphi_b = \exp(\boldsymbol{k}_b \cdot \boldsymbol{r})$ is the plane wave determining the motion of a free particle with momentum $\boldsymbol{p}_b = \hbar\boldsymbol{k}_b$, we can rewrite (95.10) as follows

$$A_{ba} = -\frac{\mu}{2\pi\hbar^2}\langle\varphi_b|\hat{V}|\psi_a\rangle. \tag{95.11}$$

The function A_{ba} is called the *scattering amplitude*. We see from (95.11) that the scattering amplitude is proportional to the reduced mass and that it depends on the energy of the relative motion, on the angle between the vectors $\boldsymbol{k}_a$ and $\boldsymbol{k}_b$, and on the scattering potential. It follows from (95.9) that at large distances from the scattering centre the wave $\psi_{sc} = A_{ba}e^{ikr}/r$ completely determines the scattering amplitude A_{ba}.

The scattering is usually characterised by the *differential scattering cross-section* $d\sigma(\theta, \varphi)$ which determines the ratio of the number of particles scattered per unit time into an element of solid angle $d\Omega = \sin\theta\, d\theta\, d\varphi$ to the flux density of incoming particles. Per second, $j_r r^2\, d\Omega$ particles will pass through an element of area $r^2\, d\Omega$, where the radial flux density j_r is given by

$$j_r = \frac{\hbar}{2\mu i}\left[\psi_{sc}^*\frac{\partial\psi_{sc}}{\partial r} - \psi_{sc}\frac{\partial\psi_{sc}^*}{\partial r}\right] = \frac{\hbar k}{\mu r^2}|A_{ba}(\theta,\varphi)|^2.$$

Using (95.4), we find, therefore, the following connexion between the differential scattering cross-section and the scattering amplitude

$$d\sigma = \frac{j_r r^2\, d\Omega}{|j_a|} = \frac{k}{k_a}|A_{ba}|^2\, d\Omega; \tag{95.12}$$

for the case of elastic scattering, $k = k_a$.