

# An example: scattering of total spin half

## 1 Connection between 3D and partial wave technique

For  $s = \frac{1}{2}$ , we define the three-dimensional (3D) basis state  $|\mathbf{p}\lambda\rangle$  as

$$|\mathbf{p}\lambda\rangle \equiv |\mathbf{p}; \frac{1}{2}\lambda\rangle = |\mathbf{p}\rangle | \frac{1}{2}\lambda\rangle. \quad (1)$$

The 3D basis state  $|\mathbf{p}\lambda\rangle$  is expanded in the partial-wave (PW) basis states  $|p(l\frac{1}{2})jm_j\rangle$  as (see former lecture notes)

$$|\mathbf{p}\lambda\rangle = \sum_{jlm_j} C\left(l\frac{1}{2}j; (m_j - \lambda)\lambda\right) \left| p\left(l\frac{1}{2}\right) jm_j \right\rangle Y_{l,m_j-\lambda}^*(\theta, \phi). \quad (2)$$

The reversal of Eq. (2), which is the expansion of the PW basis states  $|p(l\frac{1}{2})jm_j\rangle$  in the 3D basis state  $|\mathbf{p}\lambda\rangle$ , is obtained as (see former lecture notes)

$$\left| p\left(l\frac{1}{2}\right) jm_j \right\rangle = \sum_{\lambda} C\left(l\frac{1}{2}j; (m_j - \lambda)\lambda\right) \int d\hat{\mathbf{p}} |\mathbf{p}\lambda\rangle Y_{l,m_j-\lambda}(\theta, \phi). \quad (3)$$

In the following we show that Eq. (3) can also be obtained by means of the orthogonality of the spherical harmonic (SH) functions

$$\int d\hat{\mathbf{p}} Y_{l'm'}^*(\theta, \phi) Y_{lm}(\theta, \phi) = 2\pi \delta_{m'm} \int_{-1}^1 d\cos\theta Y_{l'm'}^*(\theta, 0) Y_{lm}(\theta, 0) = \delta_{l'l} \delta_{m'm} \quad (4)$$

and of the Clebsch-Gordan (CG) coefficients

$$\sum_{m_2} C(j_1 j_2 j'_3; (m_3 - m_2)m_2) C(j_1 j_2 j_3; (m_3 - m_2)m_2) = \delta_{j'_3 j_3}. \quad (5)$$

We multiply Eq. (2) with  $Y_{l,m_j-\lambda}(\theta, \phi)$  and perform an integral over  $\hat{\mathbf{p}}$  as

$$\begin{aligned} \int d\hat{\mathbf{p}} |\mathbf{p}\lambda\rangle Y_{l,m_j-\lambda}(\theta, \phi) &= \sum_{j'l'm'_j} C\left(l'\frac{1}{2}j'; (m'_j - \lambda)\lambda\right) \left| p\left(l'\frac{1}{2}\right) j'm'_j \right\rangle \int d\hat{\mathbf{p}} Y_{l',m'_j-\lambda}^*(\theta, \phi) Y_{l,m_j-\lambda}(\theta, \phi) \\ &= \sum_{j'l'm'_j} C\left(l'\frac{1}{2}j'; (m'_j - \lambda)\lambda\right) \left| p\left(l'\frac{1}{2}\right) j'm'_j \right\rangle \delta_{l'l} \delta_{m'_j m_j} \\ &= \sum_{j'} C\left(l\frac{1}{2}j'; (m_j - \lambda)\lambda\right) \left| p\left(l\frac{1}{2}\right) j'm_j \right\rangle. \end{aligned} \quad (6)$$

Next we multiply Eq. (6) with  $C(l\frac{1}{2}j; (m_j - \lambda)\lambda)$  and sum over  $\lambda$  as

$$\begin{aligned}
& \sum_{\lambda} C\left(l\frac{1}{2}j; (m_j - \lambda)\lambda\right) \int d\hat{\mathbf{p}} |\mathbf{p}\lambda\rangle Y_{l,m_j-\lambda}(\theta, \phi) \\
&= \sum_{j'} \left| p\left(l\frac{1}{2}\right) j'm_j \right\rangle \sum_{\lambda} C\left(l\frac{1}{2}j; (m_j - \lambda)\lambda\right) C\left(l\frac{1}{2}j'; (m_j - \lambda)\lambda\right) \\
&= \sum_{j'} \left| p\left(l\frac{1}{2}\right) j'm_j \right\rangle \delta_{j'j} \\
&= \left| p\left(l\frac{1}{2}\right) jm_j \right\rangle.
\end{aligned} \tag{7}$$

Thus, we get

$$\left| p\left(l\frac{1}{2}\right) jm_j \right\rangle = \sum_{\lambda} C\left(l\frac{1}{2}j; (m_j - \lambda)\lambda\right) \int d\hat{\mathbf{p}} |\mathbf{p}\lambda\rangle Y_{l,m_j-\lambda}(\theta, \phi). \tag{8}$$

As already shown in former lecture notes, to calculate observables we need the T-matrix elements  $T_{\lambda'\lambda}(p', p, \theta')$ . Now we connect  $T_{\lambda'\lambda}(p', p, \theta')$  to the T-matrix elements in the PW basis states defined as (applying conservation of total angular momentum and spin)

$$T_{l'l}^{js}(p', p) \equiv \langle p' (l's) jm_j | T | p (ls) jm_j \rangle. \tag{9}$$

Using Eq. (2) we obtain

$$\begin{aligned}
T_{\lambda'\lambda}(p', p, \theta') &= e^{-i(\lambda-\lambda')\phi'} T_{\lambda'\lambda}(\mathbf{p}', p\hat{\mathbf{z}}) \\
&= e^{-i(\lambda-\lambda')\phi'} \langle \mathbf{p}'\lambda' | T | p\hat{\mathbf{z}}\lambda \rangle \\
&= e^{-i(\lambda-\lambda')\phi'} \sum_{j'l'm'_j} \sum_{jlm_j} C\left(l'\frac{1}{2}j'; (m'_j - \lambda')\lambda'\right) C\left(l\frac{1}{2}j; (m_j - \lambda)\lambda\right) \\
&\quad \times Y_{l',m'_j-\lambda'}(\theta', \phi') Y_{l,m_j-\lambda}^*(\hat{\mathbf{z}}) \left\langle p'\left(l'\frac{1}{2}\right) j'm'_j \middle| T \middle| p\left(l\frac{1}{2}\right) jm_j \right\rangle \\
&= e^{-i(\lambda-\lambda')\phi'} \sum_{j'l'm'_j} \sum_{jlm_j} C\left(l'\frac{1}{2}j'; (m'_j - \lambda')\lambda'\right) C\left(l\frac{1}{2}j; (m_j - \lambda)\lambda\right) \\
&\quad \times Y_{l',m'_j-\lambda'}(\theta', \phi') \delta_{m_j-\lambda, 0} \sqrt{\frac{2l+1}{4\pi}} T_{l'l}^{j\frac{1}{2}}(p', p) \delta_{j'j} \delta_{m'_jm_j} \\
&= e^{-i(\lambda-\lambda')\phi'} \sum_{j'l'} \sqrt{\frac{2l+1}{4\pi}} T_{l'l}^{j\frac{1}{2}}(p', p) C\left(l'\frac{1}{2}j; (\lambda - \lambda')\lambda'\right) C\left(l\frac{1}{2}j; 0\lambda\right) \\
&\quad \times Y_{l',\lambda-\lambda'}(\theta', \phi'). \tag{10}
\end{aligned}$$

The SH functions  $Y_{lm}(\theta, \phi)$  depend on the azimuthal angle  $\phi$  as

$$Y_{lm}(\theta, \phi) = e^{im\phi} Y_{lm}(\theta, 0). \tag{11}$$

Hence, Eq. (10) now reads

$$\begin{aligned} T_{\lambda'\lambda}(p', p, \theta') &= e^{-i(\lambda-\lambda')\phi'} \sum_{jl'l} \sqrt{\frac{2l+1}{4\pi}} T_{l'l}^{j\frac{1}{2}}(p', p) C\left(l'\frac{1}{2}j; (\lambda-\lambda')\lambda'\right) C\left(l\frac{1}{2}j; 0\lambda\right) \\ &\quad \times e^{i(\lambda-\lambda')\phi'} Y_{l',\lambda-\lambda'}(\theta', 0) \\ &= \sum_{jl'l} \sqrt{\frac{2l+1}{4\pi}} T_{l'l}^{j\frac{1}{2}}(p', p) C\left(l'\frac{1}{2}j; (\lambda-\lambda')\lambda'\right) C\left(l\frac{1}{2}j; 0\lambda\right) Y_{l',\lambda-\lambda'}(\theta', 0). \end{aligned} \quad (12)$$

Finally, for the system being considered  $j = \text{odd} \times \frac{1}{2}$  and  $l', l = j \pm \frac{1}{2}$ . This leads to  $|l' - l| = 0$  or  $|l' - l| = 1$ , but parity conservation allows only  $|l' - l| = 0$ . Therefore, we have

$$T_{\lambda'\lambda}(p', p, \theta') = \sum_{jl} \sqrt{\frac{2l+1}{4\pi}} T_{ll}^{j\frac{1}{2}}(p', p) C\left(l\frac{1}{2}j; (\lambda-\lambda')\lambda'\right) C\left(l\frac{1}{2}j; 0\lambda\right) Y_{l,\lambda-\lambda'}(\theta', 0). \quad (13)$$

The T-matrix elements  $T_{\lambda'\lambda}(p', p, \theta')$  obey the following symmetry relation:

$$T_{\lambda'\lambda}(p', p, \theta') = (-)^{\lambda'-\lambda} T_{-\lambda',-\lambda}(p', p, \theta'). \quad (14)$$

Hence, we need only  $T_{\frac{1}{2}\frac{1}{2}}(p', p, \theta')$  and  $T_{-\frac{1}{2}\frac{1}{2}}(p', p, \theta')$ , which in terms of  $T_{ll}^{j\frac{1}{2}}(p', p)$  read

$$T_{\frac{1}{2}\frac{1}{2}}(p', p, \theta') = \sum_{jl} \sqrt{\frac{2l+1}{4\pi}} T_{ll}^{j\frac{1}{2}}(p', p) C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) Y_{l0}(\theta', 0) \quad (15)$$

$$T_{-\frac{1}{2}\frac{1}{2}}(p', p, \theta') = \sum_{jl} \sqrt{\frac{2l+1}{4\pi}} T_{ll}^{j\frac{1}{2}}(p', p) C\left(l\frac{1}{2}j; 1, -\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) Y_{l1}(\theta', 0). \quad (16)$$

The reversal relation of Eq. (13) is obtained by means of the orthogonality of the SH functions and of the CG coefficients as follows. We multiply Eq. (13) with  $Y_{l',\lambda-\lambda'}^*(\theta', 0)$  and perform integral over  $\cos \theta'$ .

$$\begin{aligned} &\int_{-1}^1 d \cos \theta' Y_{l',\lambda-\lambda'}^*(\theta', 0) T_{\lambda'\lambda}(p', p, \theta') \\ &= \sum_{jl} \sqrt{\frac{2l+1}{4\pi}} T_{ll}^{j\frac{1}{2}}(p', p) C\left(l\frac{1}{2}j; (\lambda-\lambda')\lambda'\right) C\left(l\frac{1}{2}j; 0\lambda\right) \int_{-1}^1 d \cos \theta' Y_{l',\lambda-\lambda'}^*(\theta', 0) Y_{l,\lambda-\lambda'}(\theta', 0) \\ &= \sum_{jl} \sqrt{\frac{2l+1}{4\pi}} T_{ll}^{j\frac{1}{2}}(p', p) C\left(l\frac{1}{2}j; (\lambda-\lambda')\lambda'\right) C\left(l\frac{1}{2}j; 0\lambda\right) \frac{\delta_{l'l}}{2\pi} \\ &= \frac{1}{4\pi} \sqrt{\frac{2l'+1}{\pi}} \sum_j T_{l'l}^{j\frac{1}{2}}(p', p) C\left(l'\frac{1}{2}j; (\lambda-\lambda')\lambda'\right) C\left(l'\frac{1}{2}j; 0\lambda\right). \end{aligned} \quad (17)$$

Next we multiply with  $C\left(l'\frac{1}{2}j'; (\lambda-\lambda')\lambda'\right)$  and sum over  $\lambda'$ .

$$\sum_{\lambda'} C\left(l'\frac{1}{2}j'; (\lambda-\lambda')\lambda'\right) \int_{-1}^1 d \cos \theta' Y_{l',\lambda-\lambda'}^*(\theta', 0) T_{\lambda'\lambda}(p', p, \theta')$$

$$\begin{aligned}
&= \frac{1}{4\pi} \sqrt{\frac{2l'+1}{\pi}} \sum_j T_{ll'}^{j\frac{1}{2}}(p', p) C\left(l'\frac{1}{2}j; 0\lambda\right) \\
&\quad \times \sum_{\lambda'} C\left(l'\frac{1}{2}j'; (\lambda - \lambda')\lambda'\right) C\left(l'\frac{1}{2}j; (\lambda - \lambda')\lambda'\right) \\
&= \frac{1}{4\pi} \sqrt{\frac{2l'+1}{\pi}} \sum_j T_{ll'}^{j\frac{1}{2}}(p', p) C\left(l'\frac{1}{2}j; 0\lambda\right) \delta_{j'j} \\
&= \frac{1}{4\pi} \sqrt{\frac{2l'+1}{\pi}} T_{ll'}^{j'\frac{1}{2}}(p', p) C\left(l'\frac{1}{2}j'; 0\lambda\right). \tag{18}
\end{aligned}$$

Now we multiply with  $C\left(l'\frac{1}{2}j'; 0\lambda\right)$ , apply the following symmetry relation for the CG coefficients

$$C(j_1 j_2 j_3; m_1 m_2 m_3) = (-)^{j_2+m_2} \sqrt{\frac{2j_3+1}{2j_1+1}} C(j_3 j_2 j_1; -m_3, m_2, -m_1), \tag{19}$$

and sum over  $\lambda$ .

$$\begin{aligned}
&\sum_{\lambda} C\left(l'\frac{1}{2}j'; 0\lambda\right) \sum_{\lambda'} C\left(l'\frac{1}{2}j'; (\lambda - \lambda')\lambda'\right) \int_{-1}^1 d \cos \theta' Y_{l', \lambda - \lambda'}^*(\theta', 0) T_{\lambda' \lambda}(p', p, \theta') \\
&= \frac{1}{4\pi} \sqrt{\frac{2l'+1}{\pi}} T_{ll'}^{j'\frac{1}{2}}(p', p) \sum_{\lambda} C\left(l'\frac{1}{2}j'; 0\lambda\right) C\left(l'\frac{1}{2}j'; 0\lambda\right) \\
&= \frac{1}{4\pi} \sqrt{\frac{2l'+1}{\pi}} T_{ll'}^{j'\frac{1}{2}}(p', p) \sum_{\lambda} \frac{(-)^{2(\frac{1}{2}+\lambda)} (2j'+1)}{(2l'+1)} C\left(j'\frac{1}{2}l'; -\lambda, \lambda\right) C\left(j'\frac{1}{2}l'; -\lambda, \lambda\right) \\
&= \frac{2j'+1}{4\pi \sqrt{(2l'+1)\pi}} T_{ll'}^{j'\frac{1}{2}}(p', p) \sum_{\lambda} C\left(j'\frac{1}{2}l'; -\lambda, \lambda\right) C\left(j'\frac{1}{2}l'; -\lambda, \lambda\right) \\
&= \frac{2j'+1}{4\pi \sqrt{(2l'+1)\pi}} T_{ll'}^{j'\frac{1}{2}}(p', p). \tag{20}
\end{aligned}$$

Thus, after changing some notations for convenience, we get

$$\begin{aligned}
T_{ll'}^{j\frac{1}{2}}(p', p) &= \frac{4\pi \sqrt{(2l+1)\pi}}{2j+1} \sum_{\lambda' \lambda} C\left(l\frac{1}{2}j; (\lambda - \lambda')\lambda'\right) C\left(l\frac{1}{2}j; 0\lambda\right) \\
&\quad \times \int_{-1}^1 d \cos \theta' Y_{l, \lambda - \lambda'}^*(\theta', 0) T_{\lambda' \lambda}(p', p, \theta'). \tag{21}
\end{aligned}$$

Applying the symmetry relation for  $T_{\lambda' \lambda}(p', p, \theta')$  given in Eq. (14) together with

$$Y_{lm}(\theta, 0) = (-)^m Y_{l, -m}(\theta, 0) \tag{22}$$

$$C(j_1 j_2 j_3; m_1 m_2) = (-)^{(j_1+j_2-j_3)} C(j_1 j_2 j_3; -m_1, -m_2) \tag{23}$$

into Eq. (21) we have

$$T_{ll'}^{j\frac{1}{2}}(p', p) = \frac{4\pi \sqrt{(2l+1)\pi}}{2j+1}$$

$$\begin{aligned}
& \times \left\{ C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l0}^*(\theta', 0) T_{\frac{1}{2}\frac{1}{2}}(p', p, \theta') \right. \\
& + C\left(l\frac{1}{2}j; 1, -\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l1}^*(\theta', 0) T_{-\frac{1}{2}\frac{1}{2}}(p', p, \theta') \\
& + C\left(l\frac{1}{2}j; -1, \frac{1}{2}\right) C\left(l\frac{1}{2}j; 0, -\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l,-1}^*(\theta', 0) T_{\frac{1}{2}, -\frac{1}{2}}(p', p, \theta') \\
& \left. + C\left(l\frac{1}{2}j; 0, -\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0, -\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l0}^*(\theta', 0) T_{-\frac{1}{2}, -\frac{1}{2}}(p', p, \theta') \right\} \\
= & \frac{4\pi\sqrt{(2l+1)\pi}}{2j+1} \\
& \times \left[ \left\{ C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) + C\left(l\frac{1}{2}j; 0, -\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0, -\frac{1}{2}\right) \right\} \right. \\
& \quad \times \int_{-1}^1 d \cos \theta' Y_{l0}^*(\theta', 0) T_{\frac{1}{2}\frac{1}{2}}(p', p, \theta') \\
& \quad + \left\{ C\left(l\frac{1}{2}j; 1, -\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) + C\left(l\frac{1}{2}j; -1, \frac{1}{2}\right) C\left(l\frac{1}{2}j; 0, -\frac{1}{2}\right) \right\} \\
& \quad \left. \times \int_{-1}^1 d \cos \theta' Y_{l1}^*(\theta', 0) T_{-\frac{1}{2}\frac{1}{2}}(p', p, \theta') \right] \\
= & \frac{4\pi\sqrt{(2l+1)\pi}}{2j+1} \\
& \times \left[ \left\{ 1 + (-)^{2(l+\frac{1}{2}-j)} \right\} \right. \\
& \quad \times C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l0}^*(\theta', 0) T_{\frac{1}{2}\frac{1}{2}}(p', p, \theta') \\
& \quad + \left\{ 1 + (-)^{2(l+\frac{1}{2}-j)} \right\} \\
& \quad \left. \times C\left(l\frac{1}{2}j; 1, -\frac{1}{2}\right) C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l1}^*(\theta', 0) T_{-\frac{1}{2}\frac{1}{2}}(p', p, \theta') \right] \\
= & \frac{4\pi\sqrt{(2l+1)\pi}}{2j+1} \left\{ 1 + (-)^{(2l-2j+1)} \right\} C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) \\
& \times \left\{ C\left(l\frac{1}{2}j; 0\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l0}^*(\theta', 0) T_{\frac{1}{2}\frac{1}{2}}(p', p, \theta') \right.
\end{aligned}$$

$$\begin{aligned}
& + C\left(l \frac{1}{2} j; 1, -\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l1}^*(\theta', 0) T_{-\frac{1}{2} \frac{1}{2}}(p', p, \theta') \Big\} \\
& = \frac{4\pi\sqrt{(2l+1)\pi}}{2j+1} \left\{ 1 - (-)^{2(l-j)} \right\} C\left(l \frac{1}{2} j; 0 \frac{1}{2}\right) \\
& \quad \times \left\{ C\left(l \frac{1}{2} j; 0 \frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l0}^*(\theta', 0) T_{\frac{1}{2} \frac{1}{2}}(p', p, \theta') \right. \\
& \quad \left. + C\left(l \frac{1}{2} j; 1, -\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l1}^*(\theta', 0) T_{-\frac{1}{2} \frac{1}{2}}(p', p, \theta') \right\}. \tag{24}
\end{aligned}$$

Since  $j = \text{odd} \times \frac{1}{2}$ , then  $2(l - j) = \text{odd}$ . Thus, finally we have

$$\begin{aligned}
T_l^{j \frac{1}{2}}(p', p) & = \frac{8\pi\sqrt{(2l+1)\pi}}{2j+1} C\left(l \frac{1}{2} j; 0 \frac{1}{2}\right) \\
& \quad \times \left\{ C\left(l \frac{1}{2} j; 0 \frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l0}^*(\theta', 0) T_{\frac{1}{2} \frac{1}{2}}(p', p, \theta') \right. \\
& \quad \left. + C\left(l \frac{1}{2} j; 1, -\frac{1}{2}\right) \int_{-1}^1 d \cos \theta' Y_{l1}^*(\theta', 0) T_{-\frac{1}{2} \frac{1}{2}}(p', p, \theta') \right\}. \tag{25}
\end{aligned}$$

## 2 Pergeseran Fase (*Phase Shift*)

Kita mulai dengan relasi matriks S dan matriks T:

$$S = 1 - 2\pi i \delta(E_{p'} - E_p) T, \tag{26}$$

kemudian kita kerjakan elemen matriksnya dalam basis gelombang parsial, dengan spin  $s = \frac{1}{2}$ , yaitu  $|p(l \frac{1}{2}) jm_j\rangle$ . Ingat bahwa:

- matriks S bersifat on-shell,
- momentum angular total dan spin kekal / tetap,
- untuk  $s = \frac{1}{2}$ , diperoleh  $|l' - l| = 0, 1$ , namun kekekalan paritas menuntut  $|l' - l| = 0$  atau  $l' = l$ .

Kita definisikan elemen matriks T-matrix (serupa pula untuk S-matrix) sebagai berikut:

$$T_l^{j \frac{1}{2}}(p', p) = \left\langle p' \left(l \frac{1}{2}\right) jm_j \middle| T \middle| p \left(l \frac{1}{2}\right) jm_j \right\rangle. \tag{27}$$

Diperoleh:

$$\begin{aligned} \left\langle p' \left( l \frac{1}{2} \right) jm_j \middle| S \left| p \left( l \frac{1}{2} \right) jm_j \right\rangle \right\rangle &= \left\langle p' \left( l \frac{1}{2} \right) jm_j \middle| p \left( l \frac{1}{2} \right) jm_j \right\rangle \\ &\quad - 2\pi i \left\langle p' \left( l \frac{1}{2} \right) jm_j \middle| \delta(E_{p'} - E_p) T \left| p \left( l \frac{1}{2} \right) jm_j \right\rangle \right\rangle \end{aligned} \quad (28)$$

$$\begin{aligned} \rightarrow S_l^{j \frac{1}{2}}(p, p) \frac{\delta(p' - p)}{p' p} &= \frac{\delta(p' - p)}{p' p} \\ &\quad - 2\pi i \int dp'' p''^2 \left\langle p' \left( l \frac{1}{2} \right) jm_j \middle| \delta(E_{p'} - E_p) \left| p'' \left( l \frac{1}{2} \right) jm_j \right\rangle \right\rangle \\ &\quad \times \left\langle p'' \left( l \frac{1}{2} \right) jm_j \middle| T \left| p \left( l \frac{1}{2} \right) jm_j \right\rangle \right\rangle \\ &= \frac{\delta(p' - p)}{p' p} - 2\pi i \int dp'' p''^2 \left\langle p' \left( l \frac{1}{2} \right) jm_j \middle| p'' \left( l \frac{1}{2} \right) jm_j \right\rangle \delta(E_{p'} - E_p) T_l^{j \frac{1}{2}}(p'', p) \\ &= \frac{\delta(p' - p)}{p' p} - 2\pi i \int dp'' p''^2 \frac{\delta(p' - p'')}{p' p''} \delta(E_{p'} - E_p) T_l^{j \frac{1}{2}}(p'', p) \\ &= \frac{\delta(p' - p)}{p' p} - 2\pi i \delta(E_{p'} - E_p) T_l^{j \frac{1}{2}}(p', p) \\ &= \frac{\delta(p' - p)}{p' p} - 2\pi i \left( \frac{dE'}{dp'} \right)^{-1} \delta(p' - p) T_l^{j \frac{1}{2}}(p, p), \quad \left( E' = \frac{p'^2}{2\mu} \right) \\ &= \frac{\delta(p' - p)}{p' p} - 2\pi i \frac{\mu}{p'} \delta(p' - p) T_l^{j \frac{1}{2}}(p, p) \\ \rightarrow S_l^{j \frac{1}{2}}(p, p) &= 1 - i2\pi\mu p T_l^{j \frac{1}{2}}(p, p). \end{aligned} \quad (29)$$

Pergeseran fase didefinisikan sebagai berikut:

$$S_l^{j \frac{1}{2}}(p, p) = e^{2i\delta_l^{j \frac{1}{2}}}. \quad (30)$$

Dari Eqs. (30) dan (29) kita dapat menghitung pergeseran fase dari  $T_l^{j \frac{1}{2}}(p, p)$  (berikut ini kita sembunyikan  $(p, p)$ ):

$$\begin{aligned} S_l^{j \frac{1}{2}} &= e^{2i\delta_l^{j \frac{1}{2}}} = \cos 2\delta_l^{j \frac{1}{2}} + i \sin 2\delta_l^{j \frac{1}{2}} = 1 + 2\pi\mu p \operatorname{Im} T_l^{j \frac{1}{2}} - i2\pi\mu p \operatorname{Re} T_l^{j \frac{1}{2}} \\ \rightarrow \tan 2\delta_l^{j \frac{1}{2}} &= \frac{\sin 2\delta_l^{j \frac{1}{2}}}{\cos 2\delta_l^{j \frac{1}{2}}} = \frac{\operatorname{Im} S_l^{j \frac{1}{2}}}{\operatorname{Re} S_l^{j \frac{1}{2}}} = \frac{-2\pi\mu p \operatorname{Re} T_l^{j \frac{1}{2}}}{1 + 2\pi\mu p \operatorname{Im} T_l^{j \frac{1}{2}}} \\ \rightarrow \delta_l^{j \frac{1}{2}} &= \frac{1}{2} \arctan \left( \frac{-2\pi\mu p \operatorname{Re} T_l^{j \frac{1}{2}}}{1 + 2\pi\mu p \operatorname{Im} T_l^{j \frac{1}{2}}} \right). \end{aligned} \quad (31)$$